1 Anne wants to crochet a rectangular blanket with 36 square meters of yarn. If the side lengths of her blanket must be positive integers and she uses up all of her yarn in the process - how many different widths can her blanket have?

Answer: $\boxed{9}$

Solution: This problem is equivalent to finding how many integer pairs divide 36. The prime factorization of 36 is $2^2 \cdot 3^2$. This gives it $(2+1) \cdot (2+1)$ different factors, giving our answer of 9.

2 Timmy is playing a game where he is given an integer x amount of money between 1 and 100 inclusive, but then has a $x\%$ chance of losing the money. What amount of money should Timmy choose to maximize his profit?

Answer: $\boxed{50}$

Solution: This is equivalent to maximizing expected value. The expected amount of money earned is $0 \cdot \frac{x}{100} + x \cdot \frac{100-x}{100} = x\frac{100-x}{100} = -\frac{1}{100} \cdot (x^2 - 100x)$. The maximum of a parabola is at the vertex. Therefore, by completing the square we get $-\frac{1}{100}(x-50)^2+25$, which as a vertex at (50,25). Thus, Timmy should choose 50.

3 How many 2-digit primes have digits summing to a prime?

Answer: $\boxed{10}$

Solution: Note that all primes above 2 are odd. Since the tens digit and ones digit must sum to a prime, and all primes greater than two are odd, the tens digit must be even. However, we must also consider summing to two, which reveals that 11 works. Finally, by bashing all possible combinations of 20, 40, 60, 80, and 1, 3, 7, 9, we find the solutions are $23, 29, 41, 43, 47, 61, 67, 83, 89$. Thus, our answer is 10.

4 How many 4-digit integers have digits multiplying to 24?

Answer: $\boxed{64}$

Solution: The possible (unordered) lists of numbers that can be digits are $(1,1,4,6)$, $(1,1,3,8)$, $(1,2,2,6)$, $(2,2,2,3)$, and $(1,2,3,4)$. In total the number of four digit integers is $12 + 12 + 12 + 4 + 24 = 64$.

5 In right triangle $\triangle$ ABC, angle C is 90 degrees and AC = 10. If angle A is between 45 and 60 degrees, exclusive, how many integer values could side length BC possibly be?

Answer: $\boxed{7}$

Solution: When angle A is 45 degrees $BC = 10$. When A is 60 degrees BC is $10\sqrt{3}$, which is approximately 17.3. In total we have 7 integers between 10 and 17.3 exclusive.

6 The circle $(x-5)^2 + (y-3)^2 = 16$ is tangent to the line $y = x + b$. Find the sum of all possible values of b.

Answer: $\boxed{-4}$

Solution: The first point of tangency has coordinates $(5-2\sqrt{2}, 3+2\sqrt{2})$, from which we can find the first y-intercept (using 45-45-90 triangles) to be $4\sqrt{2}-2$. Drawing a trapezoid using the two tangent lines, we can find the total length between the y-intercepts to be $8\sqrt{2}$. As such, the second y-intercept is at $-4\sqrt{2}-2$. Summing these gives -4.

7 If r_1 and r_2 are roots for the quadratic x^2-7x+5, find ${r_1}^3+{r_2}^3$.

Answer: $\boxed{238}$

Solution: Note that $(r_1+r_2)^3 = r_1^3+r_2^3+3r_1r_2(r_1+r_2)$. From Vieta's we have $r_1+r_2=7$ and $r_1r_2=5$. Substituting and solving gives $r_1^3+r_2^3=238$.

8 How many lattice points lie on the interior of the figure described by the equation $(|2x|+|3y|-12)^2<36$?

Answer: $\boxed{88}$

Solution: Taking the square root and adding 12, we get $6<|2x|+|3y|<18$. Graphing gives us two concentric rhombuses, with the inner one having diagonals of length 6 and 4 and the outer one having diagonals of length 18 and 12. We can just count one quadrant, multiply by 4, and add the lattice points on the axes. Looking in quadrant one, the total number of lattice points is $4+4+3+3+2+1+1=18$. The number of lattice points on the axes is $2\cdot(3+5)=16$. As such, the total number of lattice points is $72+16=88$.

9 Find the smallest positive integer k such that $k\cdot(2^2-1)(3^2-1)(4^2-1)...(2024^2-1)$ is a perfect square.

Answer: $\boxed{253}$

Solution: Since $n^2-1=(n-1)(n+1)$, we can rewrite the expression as

$$k(2-1)(2+1)(3-1)(3+1)\dots(2024-1)(2024+1)$$

$$=k\cdot1\cdot2\cdot3^2\cdot4^2\cdot\dots\cdot2023^2\cdot2024\cdot2025$$

Since 2025 is already a perfect square, we need $k\cdot2\cdot2024$ to be a perfect square. We can factor this into $k\cdot2^4\cdot11\cdot23$, so k must have a factor of 11 and of 23. The smallest possible value of k is $11\cdot23=253$.

10 We define the distance between two vertices on a regular icosahedron to be the least number of edges to be traversed to get from one point to the other. An ant is at vertex A at time 0. Every second, the ant chooses a random vertex adjacent to its current vertex (one edge away) and moves to it. What is the probability the ant will be at a distance of 2 from vertex A after 4 seconds?

Answer: $\boxed{\frac{52}{125}}$

Solution: Orient the icosahedron so that vertex A is at the top, and there are 4 distinct heights for the vertices. The ant could either move down 3 times and up once, or move down twice and horizontally twice. If the ant moves down 3 then up 1, there are 5 options for its first move, 2 for its second move, 1 for its third move, and 5 for its last move, for a total of 50 paths. If the ant moves down 2, up 1, down 1, there are $5 \cdot 2 \cdot 2 \cdot 2 = 40$ paths. If the ant moves down 1, up 1, down 2, there are $5 \cdot 1 \cdot 5 \cdot 2 = 50$ paths. If the ant moves down 2 and horizontally 2 in any order, it has 5 options the first time it moves down, 2 options the second time it moves down, and 2 ways to move horizontally each time. Since the ants first move must be moving down, there are 3 valid ways to arrange 2 downward moves and 2 horizontal moves, so there are a total of 120 paths in this case. Overall, there are 260 paths where the ant ends at A, and $5^4 = 625$ total paths, so the probability is $\frac{260}{625} = \frac{52}{125}$.

11 Define an operation $x \,\&\, y = x + y^2$. What is $((...(((1 \,\&\, 3) \,\&\, 5) \,\&\, ...) \,\&\, 19)$?

Answer: $\boxed{1330}$

Solution: Note that $(x \,\&\, y) \,\&\, z = (x + y^2) \,\&\, z = x + y^2 + z^2$. Therefore,

$$((...(((1 \,\&\, 3) \,\&\, 5) \,\&\, ...) \,\&\, 19) = 1 + 3^2 + 5^2 + \ldots + 19^2$$
$$= \left(1^2 + 2^2 + 3^2 + \ldots + 19^2\right) - \left(2^2 + 4^2 + \ldots + 18^2\right)$$
$$= \left(1^2 + 2^2 + 3^2 + \ldots + 19^2\right) - 4\left(1^2 + 2^2 + \ldots + 9^2\right)$$
$$\frac{19 \cdot 20 \cdot 39}{6} - 4 \cdot \frac{9 \cdot 10 \cdot 19}{6}$$
$$= 1330$$

12 Let A be the greatest lower bound of the smallest possible angle in an equilateral, convex pentagon. Find sin(A).

Answer: $\boxed{\dfrac{\sqrt{15}}{8}}$

Solution: The smallest possible angle appears when two pairs disjoint pairs of adjacent sides have 180° angles between them, so the pentagon looks like an isosceles triangle. Then, A is the vertex angle of that isosceles triangle. Wlog assume the triangle has sides of length 1, 2, and 2. By Law of Cosines, $\cos A = \frac{7}{8}$, so

$$\sin A = \sqrt{1 - \frac{49}{64}} = \frac{\sqrt{15}}{8}.$$

13 Given a hexagon, how many ways are there to paint all its corners either red or blue? Rotations and reflections count as identical hexagons.

Answer: $\boxed{13}$

Solution: We can do casework on the number of blue corners. If there are 0 or 1 blue corners, clearly there is only 1 hexagon each. If there are 2 blue corners, they can be adjacent, opposite, or have 1 vertex in between, for a total of 3 possibilities.

Having 4, 5, or 6 blue vertices are symmetric to these three cases. If there are 3 blue vertices and 3 red vertices, the blue vertices can be consecutive, form a right triangle, or be equally spaced and form an equilateral triangle. Therefore, there are a total of $2(1+1+3)+3 = 13$ hexagons.

14 Evaluate $9^{9^{9!}} \pmod{5!}$

Answer: $\boxed{9}$

Solution: We can prime factorize 5! as $2^3 \cdot 3 \cdot 5$ to use the Chinese Remainder Theorem. Since $9 \equiv 1 \pmod 8$, $9^{9^{9!}} \equiv 1 \pmod 8$. Since 9 is a multiple of 3, $9^{9^{9!}} \equiv 0 \pmod 3$. Since $9 \equiv -1 \pmod 5$ and $9^{9!}$ is odd, $9^{9^{9!}} \equiv -1 \equiv 4 \pmod 5$. Therefore, $9^{9^{9!}} \equiv 9 \pmod{120}$.

15 In triangle ABC, $AB = 5$, $AC = 6$, and $BC = 7$. There exists a point P for which the distance from P to AB is 2 and the distance from P to AC is 3. Find the sum of all possible distances from P to BC.

Answer: $\boxed{\dfrac{48\sqrt{6}}{7}}$

Solution: Let ℓ_1 and ℓ_2 be the two lines a distance of 2 from AB, and let m_1 and m_2 be the two lines a distance of 3 from AC. P could be any of the 4 intersections of ℓ_1, ℓ_2, m_1, and m_2. Note that the 4 lines form a parallelogram with center A, so the average height from the 4 vertices to BC is the height from A to BC.
By Heron's formula, we have $[ABC] = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}$. Therefore, the height from A to BC is $\frac{12\sqrt{6}}{7}$, so the sum of the 4 distances is $\frac{48\sqrt{6}}{7}$.