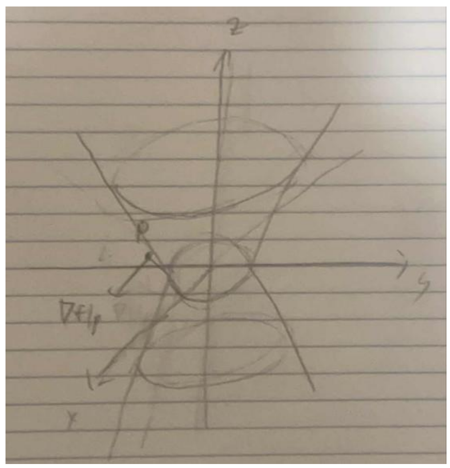
Solutions to Gödel Number Theory

1 The seniors at Montgomery Blair High School are going on a field trip. There will be 200 students and 25 teachers on the trip. Each bus can carry 45 passengers. How many buses will be needed?

Answer: $\boxed{5}$

Solution: There are a total of 200+25=225 people, and since each bus has 45 people, we need $\frac{225}{45} = 5$ buses.

2



A number is called relatively prime to another number if they share no factors other than 1. How many positive integers less than 23 are relatively prime to 23?

Answer: $\boxed{22}$

Solution: Since 23 is prime, all numbers that are not multiples of 23 and relatively prime to 23. Furthermore, no positive integer less than 23 can be a multiple of 23, so all positive integers less than 23 are relatively prime to 23. There are 22 positive integers less than 23.

3 If x, y are nonnegative integers, and $xy + x + 3y = 1$, find $x + y$.

Answer: $\boxed{1}$

Solution: If $y \geq 1$, then $xy \geq 0$ and $3y \geq 3$, so $xy + x + 3y \geq 3 \neq 1$. Therefore, $y < 1$, so $y = 0$. This gives us $x = 1$, so $x + y = 1$.

4 A positive integer is "inspirational" if it has at least three factors and the sum of its three smallest positive factors is 12. How many inspirational numbers are less than 2024?

Answer: $\boxed{0}$

Solution: The smallest factor of any number is 1, so the two next smallest factors must sum to 11. We can check each pair of positive integers that add to 11.

- 2 and 9 don't work because if 9 is a factor, 3 must be a factor.
- 3 and 8 don't work because if 8 is a factor, 2 must be a factor
- 4 and 7 don't work because if 4 is a factor, 2 must be a factor
- 5 and 6 don't work because if 6 is a factor, 2 must be a factor

Therefore, it's impossible to have the 3 smallest factors of a number sum to 12.

5 Let x equal $16^2 + 2^{16} + 4^4 + 1$, find the greatest prime factor of x.

Answer: $\boxed{257}$

Solution: We can rewrite the expression as

$$\begin{aligned} x &= \left(2^4\right)^2 + 2^{16} + \left(2^2\right)^4 + 1 \\ &= 2^{16} + 2 \cdot 2^8 + 1 \\ &= \left(2^8 + 1\right)^2 = 257^2. \end{aligned}$$

Since 257 is prime, the largest prime divisor of x is 257.

6 What is the sum of positive integers less than 81 that do not have a "2" when expressed in base 3?

Answer: $\boxed{320}$

Solution: A number less than 81 has 4 digits is base 3, any of which could be 0. We can consider the contribution of each digit to the sum individually. If the first digit is 0, it doesn't contribute anything to the sum. For any number whose first digit is 1, the first digit contributes 27 to the sum, and there are 8 ways to choose the remaining 3 digits so that each is either 0 or 1, so there are 8 such numbers. Therefore, across all numbers, the first digit contributes $27 \cdot 8$ to the sum. We can apply the same logic to the remaining 3 digits, to get that the sum is $(27 + 9 + 3 + 1) \cdot 8 = 320$.

7 What is the remainder when the product of the first 2024 prime numbers is divided by 1012?

Answer: $\boxed{506}$

Solution: We can prime factorize 1012 into $2^2 \cdot 11 \cdot 23$. Let P be the product of the first 2024 prime numbers. Then, P is a multiple of 2, 11, and 23, but not 4. By the Chinese Remainder Theorem, there is a unique value of P modulo 1012. We can see that $2 \cdot 11 \cdot 23 = 506$ satisfies all the conditions, so $P \equiv 506 \pmod{1012}$.

8 Evaluate $13^{11^{7^{5^{3^2}}}}$ mod 17.

Answer: $\boxed{4}$

Solution: We know 5^{3^2} is odd, so let $2a+1 = 5^{3^2}$. Then,

$$7^{2a+1} \equiv 49^a \cdot 7 \equiv 1^a \cdot 3 \equiv 3 \pmod 4.$$

Let $4b+3 = 7^{2a+1}$. Then,

$$11^{4b+3} \equiv 3^{4b+3} \equiv 81^b \cdot 27 \equiv 1^b \cdot 3 \equiv 3 \pmod 4.$$

Let $4c+3 = 11^{4b+3}$. Then,

$$13^{4c+3} \equiv (-4)^{4c} \cdot (-4)^3 \equiv 16^{2c} \cdot 4 \equiv 1^c \cdot 4 \equiv 4 \pmod{17}.$$