1 Jenny bought 30 of mystery boxes and received 6 large prizes. If the probability of winning a large prize stays the same, how many large prizes is she expected to win from 20 mystery boxes?

Answer: $\boxed{4}$

Solution: The probability of receiving a large prize is $\frac{6}{30} = \frac{1}{5}$. Thus, the expected number of large prizes from opening 20 boxes is $\frac{1}{5} \cdot 20 = 4$.

2 Every day, Mr. Schwartz has a 30% chance of chucking a ball of paper at a sleeping student. What is the probability that he doesn't throw paper balls for 2 consecutive days? Express your answer as a common fraction.

Answer: $\boxed{\dfrac{49}{100}}$

Solution: The probability that he does NOT through a ball on each day is $1 - \frac{3}{10} = \frac{7}{10}$. Then the probability Mr. Schwartz does not throw a ball two days in a row is $(\frac{7}{10})^2 = \frac{49}{100}$.

3 Jason has a fair coin. Evan has a weighted coin where the probability of landing heads is $\frac{3}{4}$. What is the minimum number of flips such that the probability that Evan lands tails for exactly one of his flips is more than twice the probability that Jason lands tails for exactly one of his flips?

Answer: $\boxed{5}$

Solution: Let's say we flip n times. Then the probability Jason lands tails exactly once is $n \cdot (\frac{1}{2})^n$, and the probability Evan lands tails exactly once is $n \cdot (\frac{3}{4})^{n-1} \cdot \frac{1}{4}$. The inequality then becomes $n \cdot (\frac{3}{4})^{n-1} \cdot \frac{1}{4} > 2 \cdot n \cdot (\frac{1}{2})^n$. Simplifying gives

$$(\frac{3}{4})^{n-1} > 4 \cdot (\frac{1}{2})^{n-1} \implies (\frac{3}{2})^{n-1} > 4$$

Testing values gives $n = 5$.

4 Lewis is learning to spell. He writes out all of the arrangements of the letters in his name and orders them alphabetically. For example, the first arrangement would be "eilsw". What position is the word "lewis" in?

Answer: $\boxed{53}$

Solution: The ordering of the letters alphabetically is "e", "i", "l", "s", and "w". All words that start with "e" and "i" are before "lewis"; there are a total of $4! \cdot 2 = 48$ such words. Upon looking at words starting with "l", all words starting with "lei" and "les" are before "lewis", for a total of $2! \cdot 2 = 4$ such words. We're now at "lew", and there are no other words before "lewis" starting with "lew". As such, there are a total of $48 + 4 = 52$ words before "lewis", so "lewis" is the 53rd word.

5 Yunyi has 25 pieces of cheese. He wants to split some (perhaps all) of the cheese between him and his two friends such that each person gets at least 5 pieces. In how many ways can this be done?

Answer: $\boxed{286}$

Solution: We will use stars and bars. First, distribute 5 pieces of cheese to everyone to satisfy the criterion that each person gets at least 5 pieces; now Yunyi has $25 - 3 \cdot 5 = 10$ pieces of cheese left over. Now imagine we get 3 extra cheeses and then randomly pick 3 cheeses as "dividers"; everything left of the first divider belongs to Yunyi, everything between the first and second dividers belongs to friend 1, and everything to the right of the second divider belongs to friend 2 (note that the dividers themselves are not distributed to the friends). Picking any three cheeses to be dividers will correspond to exactly one way of splitting the remaining 10 pieces of cheese. As such, in total we have 13 pieces of cheese (after adding dividers) and we wish to pick 3 of them as dividers, for a total of $\binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} = 286$.

6 Nicholas likes words because they make sense. How many rearrangements of the letters in the word "senselessness" contain the word "sense" somewhere within it?

Answer: $\boxed{7410}$

Solution: To solve this problem, we'll first count the total number of times "sense" shows up and subtract the double counts caused by words containing multiple repetitions of "sense".
In total there are 6 "s"'s, 4 "e"'s, 2 "n"'s, and 1 "l" in "senselessness", for a total of 13 letters. The number of different ways "sense" can be placed in these 13 letters is 9. After placing down "sense", we have 4 "s"'s, 2 "e"'s, 1 "n", and 1 "l" remaining to place in 8 available spots. Thus, the total number of times "sense" shows up in all counts is $9 \cdot \binom{8}{4} \cdot \binom{4}{2} \cdot 2 = 9 \cdot 70 \cdot 6 \cdot 2 = 7560$.
Now we will worry about the overcount. There are two main categories for these overcounts; either two separate showings of "sense" or a single "sensense". Both of these overcount once so we must subtract each overcount once. We can be reassured that there will be no further overcounts because there are only two "n"'s so it's impossible to have a third "sense".
For the first case we have $4 + 3 + 2 + 1 = 10$ total ways to place down the two "sense"'s. The remaining letters are "s", "s", and "l", with a total of 3 ways to arrange them. The first case thus gives us a total of $10 \cdot 3 = 30$ overcounts.
For the second case, there are 6 ways to place down "sensense". The remaining letters are "s", "s", "s", "e", and "l". The number of ways to rearrange these letters is $\binom{5}{2} \cdot 2 \cdot 1 = 20$. As such, this gives us $6 \cdot 20 = 120$ overcounts.
In total, we have $7560 - 30 - 120 = 7410$ words containing "sense".

7 You have a die that can roll any integer between 1 and 20 inclusive with equal probability. You want to design another custom die that can roll any integer between 1 and x inclusive with equal probability. Which positive integer x should you choose to maximize the probability that, when you independently roll your two dice, their values sum to 24?

Answer: $\boxed{23}$

Solution: Let $P(n)$ be the probability of rolling an n on the other die. Then the probability of obtaining a sum of 24 is $\frac{1}{20} \cdot P(23) + \frac{1}{20} \cdot P(22) + ... + \frac{1}{20} \cdot P(4)$. As such, we must try to maximize $P(4) + P(5) + ... + P(23)$.

First of all, it's clear that $x < 24$, as for $x > 23$ every one of $P(4)$ through $P(23)$ will be less than that for if $x = 23$. If $x < 24$, the probability of rolling $P(n)$ is $\frac{1}{x}$ up to $n = x$, after which $P(n) = 0$. Thus, the total value for $P(4) + P(5) + ... + P(23)$ is $\frac{x-3}{x} = 1 - \frac{3}{x}$. Note that this maximizes as x grows larger, but as we found earlier x is capped at 23. As such, $x = 23$ maximizes $P(4) + P(5) + ... + P(23)$ and subsequently the probability of obtaining a sum of 24.

8 A sequence of 7 digits is "pandemic" if it consists entirely of the digits 0 and 2, and it's possible to remove 3 digits so that the remaining 4 read 2020 in order. For example, 2022200 is pandemic because we can remove the 4th, 5th, 6th digits from the left so that the remaining digits form 2020, but 2200000 is not pandemic. How many pandemic strings are there?

Answer: $\boxed{64}$

Solution: We will do casework on the number of 0's and the number of 2's. Note that we need at least two 0's and two 2's.

Case 1: There are two 2's and five 0's
Our sequence will be in the form $A2B2C$, where A, B, and C are sequences of zeros. For this sequence to be pandemic, B and C must each contain a 0. So, we need to distribute the three remaining 0's among A, B, and C, which, from stars and bars, is $\binom{5}{2} = 10$.

Case 2: There are three 2's and 4 0's.
Our sequence will be in the form $A2B2C2D$, where A, B, C, and D are sequences of zeros. If B and C each contain a 0, we can distribute the remaining two 0's in $\binom{5}{3} = 10$ ways. If B and D each contain a 0, we can again distribute the remaining two 0's in $\binom{5}{3}$ ways. Finally, if C and D each contain a 0, we can again distribute the remaining two 0's in $\binom{5}{3}$ ways. However, if B, C, and D each contain a 0, we've counted the sequence thrice. There are $\binom{4}{1} = 4$ ways for this to happen, so we must subtract $2 \cdot 4 = 8$ sequences to correct for overcounting. So, there's a total of $3 \cdot 10 - 8 = 22$ sequences in this case.

Case 3: There are four 2's and three 0's.
This is symmetric to case 2, giving us 22 sequences.

Case 4: There are five 2's and two 0's.
This is symmetric to case 1, giving us 10 sequences.

So, there are a total of $10 + 22 + 22 + 10 = 64$ pandemic sequences.