

## Solutions to Brahmagupta Algebra

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- 1 Carlos is piloting his drone. Each move, his drone can either move up 4 feet, or move down 2 feet. What is the least number of moves in which Carlos can move his drone up exactly 10 feet from its starting position?

**Answer:**  $\boxed{4}$

**Solution:** Carlos must move his drone up at least three times to exceed 10 feet. After he does so, moving down once brings him to exactly 10 feet, for a total of 4 moves.

- 2 You are playing Guess the Number with Mr. Schwartz. You start with 10 and that is not his number. You then add 5, multiply by  $n$ , divide by 5, and get 9, which is Mr. Schwartz's number. What is  $n$ ?

**Answer:**  $\boxed{3}$

**Solution:** Working backwards,  $9 \cdot 5 = (10 + 5) \cdot x$ , so  $x$  must be 3.

- 3 Chris and David are comparing their levels from a game, which are integers greater than 2. They find that 21 more than the sum of their levels is equal to the product of their levels. If Chris's level is odd, compute David's level.

**Answer:**  $\boxed{12}$

**Solution:** Let Chris and David's levels be  $c$  and  $d$ , respectively. Then,  $cd - c - d = 21$ . Factoring, we get  $c(d - 1) - d = 21$ , and adding one gives us  $c(d - 1) - (d - 1) = 22$ , or  $(c - 1)(d - 1) = 22$ . Then, Chris and David's levels must be either 2 and 23 or 3 and 12, but since having a level of 2 is not allowed,  $c$  and  $d$  must be 3 and 12 in some order. Chris's level is odd, so David's level is 12.

- 4 Let  $f$  be a linear function ( $f(x) = ax + b$  for some  $a$  and  $b$ ). If  $f(z) = 8$ ,  $f(f(z)) = 20$ , and  $f(f(f(z))) = 44$ , what is the value of  $z$ ?

**Answer:**  $\boxed{2}$

**Solution:** By substitution,  $f(f(z)) = f(8) = 20$  and  $f(f(f(z))) = f(20) = 44$ . The linear function that matches  $f(8) = 20$  and  $f(20) = 44$  is  $y = 2x + 4$ . To solve for  $z$ ,  $2z + 4 = 8$ , so  $z = 2$ .

- 5 Initially, on a whiteboard, the numbers 2023 and 1 are written side by side. 2023 is written to the left. During each of the next 100 days, Aiden will erase the smaller number on the whiteboard and replace it with twice the original number. By the end of 100 days, how many times will he have erased a number from the left spot?

**Answer:**  $\boxed{45}$

**Solution:** For the first 11 days, Aiden will only erase the numbers on the left. In the start of the 12th day, Aiden will have 2048 on the left and 2023 on the right, so he will erase the number of the right. Note that every day after the 12th, Aiden will alternate between left and right, so the final answer is the number of even numbers between 12 and 100, which is  $\frac{100-12}{2} + 1 = 45$ .

- 6 In an arithmetic sequence  $a_n$ , the terms  $a_1$ ,  $a_2$ , and  $a_5$  form a geometric sequence. Given that  $a_1 = 2$ , what is the largest possible value of  $a_{10}$ ? (An arithmetic sequence is a list of numbers where the difference between consecutive terms is constant. A geometric sequence is a list of numbers where the ratio between consecutive terms is constant.)

**Answer:** 38

**Solution:** Because the sequence is arithmetic, term  $a_2 = 2 + d$  for some common difference  $d$ . Likewise,  $a_5 = 2 + 4d$ . Since  $a_1$ ,  $a_2$ , and  $a_5$  form geometric sequences, it follows that  $\frac{a_2}{a_1} = \frac{a_5}{a_2}$ , so  $\frac{2+d}{2} = \frac{2+4d}{2+d}$ . Cross-multiplying, we get the quadratic  $d^2 + 4d + 4 = 8d + 4$ , or  $d^2 - 4d = 0$ . Obviously,  $a_{10}$  is maximized when  $d = 4$ , so  $a_{10} = 2 + 9 \cdot 4 = 38$ .

- 7 Find the sum of the squares of the roots to the equation  $2x^3 + 4x^2 + x + 3 = 0$ .

**Answer:** 3

**Solution:** Let the roots be  $r_1, r_2$ , and  $r_3$ . Note that  $r_1 + r_2 + r_3 = -2$  and  $r_1r_2 + r_1r_3 + r_2r_3 = \frac{1}{2}$  by Vieta's Formula. Because  $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_1r_3 + r_2r_3)$ ,  $r_1^2 + r_2^2 + r_3^2 = 4 - 1 = 3$ .

- 8 For real  $a$  and  $b$ , let  $f(x) = (2a - x)(x + 3)$ .  $f(x)$  has a maximum value of  $2b$ . Furthermore, let  $g(x) = bx^2 + 8x - 6$ , where  $g(x)$  has a vertex with y-coordinate  $-b$ . Find the maximum value of  $a^2 \cdot b$ .

**Answer:** 242

**Solution:** We can first rewrite  $g(x)$  in vertex form by completing the square:  $g(x) = b(x + \frac{4}{b})^2 - 6 - \frac{16}{b}$ . Setting the y-coordinate equal to  $-b$ , we get  $-6 - \frac{16}{b} = -b \Rightarrow b^2 - 6b - 16 = 0 \Rightarrow (b+2)(b-8) = 0 \Rightarrow b = -2, 8$ . For  $f(x)$ , the maximum value is the y-coordinate of the vertex, and the vertex's x-coordinate is the midpoint of the x-intercepts. The x-coordinate of the vertex is therefore  $\frac{2a+3}{2}$ . Plugging this into  $f(x)$ , we get  $(\frac{2a+3}{2})^2 = 2b$ . The left hand side is non-negative, so  $b = 8$ .  $(\frac{2a+3}{2})^2 = 16 \Rightarrow 2a + 3 = \pm 8 \Rightarrow a = \frac{-3 \pm 8}{2}$ . To maximize  $a^2 \cdot b$ ,  $a = -\frac{11}{2}, b = 8$ , giving us 242.