1 There are 5 red balls and 3 blue balls in a bag. Alice randomly picks a ball out of the bag and then puts it back in the bag. Bob then randomly picks a ball out of the bag. What is the probability that Alice gets a red ball and Bob gets a blue ball, assuming each ball is equally likely to be chosen?

Proposed by Valerie Song.

Answer: $\boxed{\frac{15}{64}}$

Solution: We can first calculate the probability of Alice getting a red ball. There are 5 red balls and 8 total, so the probability of Alice getting a red ball is $\frac{5}{8}$. Similarly, since Alice puts her ball back into the bag, the probability that Bob gets a blue ball is $\frac{3}{8}$. These events are independent, so the probability of both events happening is $\frac{5}{8} \cdot \frac{3}{8} = \frac{15}{64}$.

2 A circle has radius 6. A smaller circle with the same center has radius 5. What is the probability that a dart randomly placed inside the outer circle is outside the inner circle?

Proposed by Tony Song.

Answer: $\boxed{\frac{11}{36}}$

Solution: The area of the outer circle is 36π, while the area of the inner circle is 25π, so the area inside the outer circle but outside the inner circle is 11π, so the chance of landing in the desired area is $\frac{11\pi}{36\pi} = \frac{11}{36}$.

3 Alex and Jeff are playing against Max and Alan in a game of tractor with 2 standard decks of 52 cards. They take turns taking (and keeping) cards from the combined decks. At the end of the game, the 5s are worth 5 points, the 10s are worth 10 points, and the kings are worth 10 points. Given that a team needs 50 percent more points than the other to win, what is the minimal score Alan and Max need to win?

Proposed by Kevin Wu.

Answer: $\boxed{120}$

Solution: There are $5 + 10 + 10 = 25$ points in a suit, so there are $25 \cdot 4 = 100$ points in a deck and $100 \cdot 2 = 200$ points in the whole game. If Alan and Max's score is x, we need $x \geq (200 - x)\frac{3}{2}$, so then $x + \frac{3}{2}x \geq 300$, so we get $x \geq \boxed{120}$.

4 Bob has a sandwich in the shape of a rectangular prism. It has side lengths 10, 5, and 5. He cuts the sandwich along the two diagonals of a face, resulting in four pieces. What is the volume of the largest piece?

Proposed by Joshua Hsieh.

Answer: $\boxed{62.5}$

Solution: No matter how the sandwich is arranged, cutting along the two diagonals will result in four pieces with the same volume. The total volume of sandwich is $10 \cdot 5 \cdot 5 = 250$, so the volume of any given piece is $\frac{250}{4} = \frac{125}{2}$.

5 Aven makes a rectangular fence of area 96 with side lengths x and y. John makes a larger rectangular fence of area 186 with side lengths $x + 3$ and $y + 3$. What is the value of $x + y$?

Proposed by Kian Dhawan.

Answer: $\boxed{27}$

Solution: From the areas, we get the two equations $xy = 96$ and $(x + 3)(y + 3) = 186$. Expanding the second one, we get $xy+3x+3y+9 = 186$. Substituting the first equation into this, $96 + 3x + 3y + 9 = 186$. Simplifying, we get $3x + 3y = 81$ or $x + y = 27$.

6 A number is prime if it is only divisible by itself and 1. What is the largest prime number n smaller than 1000 such that $n + 2$ and $n - 2$ are also prime? Note: 1 is not prime.

Proposed by Heerok Das.

Answer: $\boxed{5}$

Solution: If n is one more than a multiple of $3, n + 2$ is divisible by 3. If n is two more than a multiple of $3, n - 2$ is divisible by 3. The only prime divisible by 3 is 3 itself, so to maximize n, we must have $n - 2 = 3$, or $n = 5$.

7 Sally has 3 red socks, 1 green sock, 2 blue socks, and 4 purple socks. What is the probability she will choose a pair of matching socks when only choosing 2 socks without replacement?

Proposed by Sophia Sun and Megan Gu.

Answer: $\boxed{\dfrac{2}{9}}$

Solution: There are 3 separate cases in which you choose a red pair, a blue pair, or purple pair. Evaluating each and summing them, we have $\frac{3}{10} \cdot \frac{2}{9} + \frac{2}{10} \cdot \frac{1}{9} + \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{9}$

8 A triangle with vertices at $(0, 0)$, $(3, 0)$, $(0, 6)$ is filled with as many 1×1 lattice squares as possible. How much of the triangle's area is not filled in by the squares?

Proposed by Kevin Yao.

Answer: $\boxed{3}$

Solution: Only 6 squares can be filled by lattice squares, so $\frac{3 \cdot 6}{2} - 6 = 9 - 6 = 3$ are not filled in.

9 Let A and B be digits. If $125A^2 + B161^2 = 11566946$ What is $A + B$?

Proposed by Sophia Sun.

Answer: $\boxed{8}$

Solution: The units digit of $125A$ squared plus the units digit of $B161$ squared is 6, so the units digit of A squared plus 1 should be 6, meaning A must be 5. To get 11566946, B must be less than 4, or the total sum would be greater than 16,000,000. It has to be greater than 2 or the sum would not be large enough, so B has to be 3. Therefore, $A + B = 8$.

10 A series of concentric circles $w_1, w_2, w_3, ...$ satisfy that the radius of $w_1 = 1$ and the radius of $w_n = \frac{3}{4}$ times the radius of w_{n-1}. The regions enclosed in w_{2n-1} but not in w_{2n} are shaded for all integers $n > 0$. What is the total area of the shaded regions?

Proposed by Yunyi Ling.

Answer: $\boxed{\dfrac{16\pi}{25}}$

Solution: $A = \pi(1 - (\frac{3}{4})^2 + (\frac{3}{4})^4 - (\frac{3}{4})^6 + (\frac{3}{4})^8 - (\frac{3}{4})^{10} + ...) = \frac{\pi}{1+\frac{9}{16}} = \frac{16\pi}{25}$

11 How many ordered pairs of integers (x, y) satisfy $y^2 - xy + x = 0$?

Proposed by Bradley Guo.

Answer: $\boxed{2}$

Solution: Solving for x, we get $x = \frac{y^2}{y-1}$. This simplifies to $x = y + \frac{1}{y-1} + 1$, which is an integer only when $y - 1|1$. Thus, there are only two possible values for y, 0 and 2. For each, there is exactly one x, so there are two ordered pairs.

12 10 cards labeled 1 through 10 lie on a table. Kevin randomly takes 3 cards and Patrick randomly takes 2 of the remaining 7 cards. What is the probability that Kevin's largest card is smaller than Patrick's largest card, and that Kevin's second-largest card is smaller than Patrick's smallest card?

Proposed by Bradley Guo.

Answer: $\boxed{\dfrac{1}{5}}$

Solution: For this to happen, Patrick must have drawn the highest card out of the 5 cards chosen and the second-highest card, or the highest card and the third-highest card. There are a total of 10 possible pairs of cards that Patrick could have chosen, so the probability is $\frac{2}{10} = \frac{1}{5}$.

13 N consecutive integers add to 27. How many possible values are there for N?

Proposed by Evan Wu.

Answer: $\boxed{8}$

Solution: The sum of n consecutive integers from a to $a+n-1$ is $(\frac{2a+n-1}{2}) \cdot n$. Thus, $\frac{n}{2}$ must be a factor of 27, so n must be a factor of 54. There are 8 possibilities for n: 1, 2, 3, 6, 9, 18, 27, and 54, each with a corresponding value for a.

14 A circle with center O and radius 7 is tangent to a pair of parallel lines l_1 and l_2. Let a third line tangent to circle O intersect l_1 and l_2 at points A and B. If $AB = 18$, find $OA + OB$.

Proposed by Bradley Guo.

Answer: $\boxed{24}$

Solution: Let the point at which AB is tangent to circle O be P. By definition of tangency, OP is perpendicular to AB. Let the tangent from l_1 to O be D and the tangent from l_2 to O be E. Since $\triangle OAD$ is similar to $\triangle OAP$, we have that OA bisects $\angle A$. Similarly, OB bisects $\angle B$. Since $\angle A + \angle B = 180$, we know that $\angle AOB = 90$. Given this, we can find $OA^2 + OB^2 = AB^2 = 324$ and $OP \cdot AB = 126 = OA \cdot OB$, where the first equation follows from the Pythagorean Theorem, and the second from the fact that both quantities are equivalent to the area of the $\triangle AOB$. We can now end the problem since

$$OA + OB = \sqrt{OA^2 + 2(OA \cdot OB) + OB^2} = \sqrt{324 + 2 \cdot 126} = \boxed{24}$$

15 Let

$$M = \prod_{i=0}^{42}(i^2 - 5).$$

Given that 43 doesn't divide M, what is the remainder when M is divided by 43?

Proposed by Kevin Wu.

Answer: $\boxed{23}$

Solution: Consider the polynomial $P(x) = \prod_{i=0}^{42}(x - i^2)$. We claim this polynomial is actually congruent to the polynomial $Q(x) = x(x^{21} - 1)^2$ when taken mod 43. To see this, notice that P is in "factored form", and both have the same leading coefficient, so it suffices that the two polynomials have the same roots. First, it is clear 0 is a root of both polynomials.

Additionally, for every nonzero $i \pmod{43}$, then $P(x)$ has the two factors $(x - i^2)(x - (-i)^2)$, so we need to show that Q has a double root at i^2 as well. We claim that $(x^{21} - 1)$ has a root at i^2. This is true because $(i^2)^{21} = i^{42} \equiv 1$ by Fermat's Little Theorem. This shows that every root of P is also a root of Q with the same multiplicity, but Q has the same degree as P, so the two polynomials must be the same. To finish, we see that

$$\prod_{i=0}^{42}(i^2-5) = -P(5) \equiv -Q(5) = -5(5^{21}-1)^2 \equiv -5(-2)^2 = -20 \equiv 23.$$

One way to compute $5^{21} \pmod{43}$ is to note that $(5^{21})^2 = 5^{42} = 1 \pmod{43}$, so it must be ± 1. However, if it were 1, then we'd get 0, so we know it must be -1.