1 Find the largest integer less than 2023 whose square ends in 9.

Proposed by Michelle Gao.

Answer: $\boxed{2017}$

Solution: Only numbers ending in 3 or 7 have squares ending in 9. The largest such integer less than 2023 is 2017.

2 How many positive integers divide both 100 and 160?

Proposed by Joshua Hsieh.

Answer: $\boxed{6}$

Solution: Any integer that divides both 100 and 160 divides their greatest common divisor, 20. There are 6 such numbers: $1, 2, 4, 5, 10, 20$.

3 There exist positive integers a, b, c, with $b > 1$, and $6 \cdot a = b \cdot c = 12000$. If a and b are relatively prime, what is c?

Proposed by Isabelle Yang.

Answer: $\boxed{4000}$

Solution: We know that $a = \frac{12000}{6} = 2000$. By prime factorizing 12000, we have that $120 = 2^5 \cdot 3^1 \cdot 5^3$ We also know that a is relatively prime to b, so b must be relatively prime with $2000 = 2^4 \cdot 5^3$. Looking at the prime factors of 12000, we see that the only way a factor of 12000 can be relatively prime to 2000 is if it is 3 or 1. But $b > 1$ so $b = 3$. Thus, we get that $c = \frac{12000}{3} = 4000$.

4 What is the largest integer n such that 3^n is a factor of $18! + 19! + 20!$?

Proposed by Kian Dhawan.

Answer: $\boxed{8}$

Solution: We first notice that each of the terms is divisible by 18! so we can factor it out. This gives us $18! + 19! + 20! = 18!(1 + 19 + 19 \cdot 20) = 18! \cdot (400)$. 400 has no factors of 3, so we just count the factors of 3 in 18!. There is 1 factor of 3 from 3, 6, 12, 15, and 2 factors of 3 from 9, 18. This adds to a total of 8 factors of 3, thus the answer is 8.

5 For some positive integer $1 \leq n \leq 1000$, Jeremy writes down n^2, n^1, and n^0 in a row on his whiteboard, in that order. His friend Joshua, however, read the three integers as a single integer and deduced that it is a multiple of 3. For how many n would this happen?

Proposed by Kelin Zhu.

Answer: $\boxed{334}$

Solution: Notice that since $10^k \equiv 1 \pmod 3$, the concatenation of n^2, n^1, n^0 is equivalent to $n^2 + n + 1$ mod 3. Testing $n = 0, 1, 2$ we find that this only works for $n \equiv 1$ mod 3. There are 334 possible solutions between 1 and 1000.

6 Suppose we have positive integers that sum up to 200. What is the largest possible product of the integers?

Proposed by Kevin Wu.

Answer: $\boxed{2 \cdot 3^{66}}$

Solution: Consider the optimal splitting of the integers. Clearly, there cannot be any integer in the split n such that $n \geq 5$, since by replacing n with $n-2, 2$ we'd do better because

$$n > 4 \implies 2n - 4 > n \implies 2(n-2) > n.$$

Additionally, we can safely replace a 4 with two 2s without changing the product.

However, notice that three 2s should be replaced by two 3s, because $2^3 < 3^2$. Therefore the number of 2s has to be less than three, so the only possibility is to have one 2 and 66 3s, giving a product of $2 \cdot 3^{66}$.

7 Find the remainder when the sum of $x(x+1)(x+2)$ for all x ranging from $x = 1$ to $x = 39$ is divided by 40.

Proposed by Bradley Guo.

Answer: $\boxed{20}$

Solution: We know that the sum of $x(x+1)(x+2)$ from 1 to 39 is equivalent to the sum of $(x-1)x(x+1) = x^3 - x$ from 2 to 40. Since we're taking this sum modulo 40, we have that the sum is equivalent to the sum of $x^3 - x$ from 1 to 39, because $x^3 - x \equiv 0 \mod 40$ for $x = 1, 40$. We can pair the terms a and $40 - a$ in the sum to get that $a^3 + (40 - a^3) - a - (40 - a) \equiv 0 \mod 40$. This leaves us with the only term that couldn't be paired: 20. Our answer is $20^3 - 20 \equiv 20 \mod 40$.

8 Find x, where x is the remainder when

$$\prod_{k=1}^{40} k!^2$$

is divided by 41.

Proposed by Kevin Wu and Nathan Cho.

Answer: $\boxed{40}$

Solution: The condition tells us that it may suffice to find two values of x that are additive inverses of each other. As 41 is a prime, this could occur when finding the solutions to an equation $a^2 \pmod{41}$. This motivates us to square our term. We wish to find

$$\prod_{k=1}^{40} k!^2 \pmod{41}.$$

Clearly, this is the same as

$$\prod_{k=1}^{40} (k^{41-k})^2 \equiv \prod_{k=1}^{20} (k^{41} \cdot (-1)^{41-k})^2 \equiv \prod_{k=1}^{20} k^{82} \pmod{41}$$

By Fermat's Little Theorem, we can see that $k^{82} \equiv k \pmod{p}$. Therefore,

Hence, it follows that our answer is 40.