1 An equilateral triangle and a square have the same perimeter. If the side length of the equilateral triangle is 8, what is the square's side length?

Proposed by Evan Zhang.

Answer: $\boxed{6}$

Solution: The perimeter of the triangle is $3 \cdot 8 = 24$. The perimeter of the square is also 24 so its side length must be $\frac{24}{4} = 6$.

2 What is the maximum possible number of sides and diagonals of equal length in a quadrilateral?

Proposed by Bradley Guo.

Answer: $\boxed{5}$

Solution: A rhombus with angles 60 and 120 degrees has 4 sides and 1 diagonal of equal length. A quadrilateral with all 4 sides and 2 diagonals of equal length is impossible because a rhombus with equal-length diagonals is a square, which has a diagonal that is longer than its side length.

3 Patrick is rafting directly across a river 20 meters across at a speed of 5 m/s. The river flows in a direction perpendicular to Patrick's direction at a rate of 12 m/s. When Patrick reaches the shore on the other end of the river, what is the total distance he has traveled?

Proposed by Kevin Yao.

Answer: $\boxed{52}$

Solution: Every second, Patrick moves 5 meters towards the other side of the river, and 12 meters along the river. Thus, he travels 13 meters every second for 4 seconds, which gives a total distance of 52.

4 Quadrilateral $ABCD$ has side lengths $AB = 7$, $BC = 15$, $CD = 20$, and $DA = 24$. It has a diagonal length of $BD = 25$. Find the measure, in degrees, of the sum of angles ABC and ADC.

Proposed by Kevin Yao.

Answer: $\boxed{180}$

Solution: 7-24-25 and 15-20-25 are both Pythagorean triples, and so two of the angles are right angles. Since the sum of the internal angles of a quadrilateral is 360, $m\angle ABC + m\angle ADC = 360 - 180 = 180$.

5 What is the largest P such that any rectangle inscribed in an equilateral triangle of side length 1 has a perimeter of at least P?

Proposed by Bradley Guo.

Answer: $\boxed{\sqrt{3}}$

Solution: Two corners of the rectangle must be on the same side of the triangle. Let the side length of the edge between the two corners be x. Then the other side length of the rectangle is $\frac{(1-x)\sqrt{3}}{2}$, and so the perimeter of the rectangle is $\sqrt{3}+(2-\sqrt{3})x$. This is minimized at $x=0$ which gives an answer of $\sqrt{3}$.

6 A circle is inscribed in an equilateral triangle with side length s. Points A, B, C, D, E, F lie on the triangle such that line segments AB, CD, and EF are parallel to a side of the triangle, and tangent to the circle. If the area of hexagon $ABCDEF = \frac{9\sqrt{3}}{2}$, find s.

Proposed by Valerie Song.

Answer: $\boxed{3\sqrt{3}}$

Solution: Consider hexagon $ABCFEF$. The sides of the equilateral triangle all make angles of 60° to each other, so the internal angles in the hexagon are all $180°-60° = 120°$.

We claim this hexagon is regular. To so this, we will prove all of its sides have the same length. L the center of the inscribed circle be I. We then examine an arbitrary pair of sides, say AB and BC. We note that I is equidistant from AB and BC, so it must lie on the angle bisector of $\angle ABC$. Therefore, $\angle ABI = \angle CBI = 60°$. Similarly, $\angle BAI = \angle BCI = 60°$, so $\triangle ABI$ and $\triangle BCI$ are equilateral, with sidelengths of BI. Thus, $AB = BI = BC$ so all pairs of adjacent sides are equal. This means all sides are equal, as desired.

Now, the area of a regular hexagon with sidelength s is $\frac{3\sqrt{3}}{2}s^2$, so equating this to $\frac{9\sqrt{3}}{2}$ gives $\sqrt{3}$. The sidelength of the larger triangle is triple this, which is $3\sqrt{3}$.

7 Let $\triangle ABC$ be such that $\angle A = 105°, \angle B = 45°, \angle C = 30°$. Let M be the midpoint of A, C. What is $\angle MBC$?

Proposed by Kevin Wu.

Answer: $\boxed{15°}$

Solution: Let E be the foot from A to BC. Then because $\triangle AEB$ is a 45-45-90 triangle, then $BE = AE$. Additionally, because $\angle AEC$ is right and M is the midpoint of A, C, then $MA = ME = MC$, and since AEC is a 30-60-90 triangle we also get $AE = AM$. Thus we get the equality $BE = AE = AM = ME = MC$.

Now, we can notice that $\triangle EMC$ is isosceles, so $\angle MEC = \angle MCE = 30°$. Finally, since $\triangle MEB$ is isosceles, then $\angle MBC = \angle MBE = \frac{\angle MEC}{2} = 15°$. Alternatively, because $AB = AE\sqrt{2}$, then we can see $\frac{AM}{AB} = \frac{1}{\sqrt{2}} = \frac{AB}{AC}$, then $\triangle AMB \sim \triangle ABC$, so $\angle ABM = 30°$, and $\angle MBC = \angle ABC - \angle ABM = 15°$.

8 Points A, B, and C lie on a circle centered at O with radius 10. Let the circumcenter of $\triangle AOC$ be P. If $AB = 16$, find the minimum value of PB.

The circumcenter of a triangle is the intersection point of the three perpendicular bisectors of the sides.

Proposed by Bradley Guo and Stephen Chen.

Answer: $\boxed{\dfrac{39}{5}}$

Solution: Fix points A, B, and O, and let C vary along the circle. We know that P must lie on the perpendicular bisector of AO, and by choosing where C lies on the circle, we can choose where P lies on the perpendicular bisector of AO. All that remains is finding the distance from B to the perpendicular bisector of AO.

Let the midpoint of AO be M, the midpoint of AB be N, and the point at which the perpendicular bisector of AO intersects AB be Q. We know that $AN = 8$, $AO = 10$, so $NO = 6$. Triangle AMQ is similar to triangle ANO, and $AM = 5$. Thus, $AQ = \frac{25}{4}$. Let the foot of the altitude from B to MQ be D. Since triangle BDQ is similar to triangle AMQ, we can finally get that $BD = \frac{39}{5}$.