1 What percent of the first 20 positive integers are divisible by 3?

Proposed by Bradley Guo.

Answer: $\boxed{30}$

Solution: A third of integers are divisible by 3. However, only 6 integers within the first 20 positive integers are divisible by 3, so the answer is $\frac{6}{20} = 30\%$.

2 It is said that a sheet of printer paper can only be folded in half 7 times. A sheet of paper is 8.5 inches by 11 inches. What is the ratio of the paper's area after it has been folded in half 7 times to its original area?

Proposed by River Chen.

Answer: $\boxed{\dfrac{1}{128}}$

Solution: Folding a paper halves one dimension, therefore halving the area. That means that seven folds halves the area 7 times, so the ratio of the paper's area after it has been folded in half 7 times to its original area is $\frac{(\frac{1}{2})^7}{1} = \frac{1}{128}$

3 Boba has an integer. They multiply the number by 8, which results in a two digit integer. Bubbles multiplies the same original number by 9 and gets a three digit integer. What was the original number?

Proposed by Bradley Guo.

Answer: $\boxed{12}$

Solution: Because multiplying the number by 8 results in a two digit number, the number must be between 2 and 12, inclusive. Furthermore, because multiplying the number by 9 results in a three digit integer, the number must be at least 12. Thus, the only number the original number could be is 12.

4 For how many integers x is $9x^2$ greater than x^4?

Proposed by Bradley Guo.

Answer: $\boxed{4}$

Solution: If $9x^2 > x^4$, we can divide x^2 (since x^2 can't be negative, we can divide it without needing to flip the inequality sign) to get $9 > x^2$. Thus, x has to be between -3 and 3, not inclusive, so there are only five integers it can be. However, if x is 0, we never could have divided by x^2, and it turns out that $9x^2$ is equal to x^4 when x is 0. That means the only four integers that work are $-2, -1, 1$, and 2.

5 How many two digit numbers are the product of two distinct prime numbers ending in the same digit?

Proposed by Kevin Yao.

Answer: $\boxed{2}$

Solution: We know that the two numbers cannot both be larger than 10, as then their product would be at least three digits. That means that one of the numbers has to be an one digit prime. Looking at the one digit primes, we can see that if the prime is 3, we can multiply it by 13 or 23 to form a two digit number, but 33 isn't prime and 43 makes the product larger than 99. If the prime is 5, no number ending in 5 greater than 5 can't be prime. If the prime is 7, the first prime number ending in the same digit larger than that prime is 17, but that product is larger than 100. As there are no other one digit primes, the only products possible are $3 \cdot 13 = 39$ and $3 \cdot 23 = 69$.

6 A triangle's area is twice its perimeter. Each side length of the triangle is doubled, and the new triangle has area 60. What is the perimeter of the new triangle?

Proposed by Bradley Guo.

Answer: $\boxed{15}$

Solution: Since the new triangle has area 60, the old triangle must have had area $\frac{60}{4} = 15$. That tells us that the old triangle had perimeter $\frac{15}{2}$, so the new one must have perimeter $\frac{15}{2} \cdot 2 = 15$

7 Let F be a point inside regular pentagon $ABCDE$ such that $\triangle FDC$ is equilateral. Find $\angle BEF$.

Proposed by Kevin Wu.

Answer: $\boxed{6}$

Solution: The angles in a regular pentagon are all 108°. Thus, $\angle EDF = \angle BCF = 108 - 60 = 48°$. Since △ FDC is equilateral, $\overline{DC} = \overline{DF} = \overline{FC}$. That means that $\overline{ED} = \overline{FD}$, so $\triangle EDF$ is isoceles. This tells us that $\angle EFD = \frac{180-48}{2} = 66$. Then, $\angle EFB = 360 - 60 - 66 - 66 = 168$. Due to symmetry, $\overline{EF} = \overline{FB}$, so $\angle BEF = \frac{180-168}{2} = 6°$.

8 Carl, Max, Zach, and Amelia sit in a row with 5 seats. If Amelia insists on sitting next to the empty seat, how many ways can they be seated?

Proposed by Bradley Guo.

Answer: $\boxed{48}$

Solution: Notice that Amelia and the chair must always be together, so we can think of Amelia and the empty chair as one being, Chairmelia. If we seat Chairmelia, Carl, Max, and Zach, there are $4! = 24$ ways to arrange them. However, the arrangement of the chair and Amelia can happen in two ways, with the the chair being left of Amelia or right of Amelia. Thus, there are $24 \cdot 2 = 48$ ways.

9 The numbers $1, 2, ..., 29, 30$ are written on a whiteboard. Gumbo circles a bunch of numbers such that for any two numbers he circles, the greatest common divisor of the two numbers is the same as the greatest common divisor of all the numbers he circled. Gabi then does the same. After this, what is the least possible number of uncircled numbers?

Proposed by Nathan Cho.

Answer: $\boxed{12}$

Solution: Call a set S *good* if the pairwise gcds of numbers in S is constant.

We note that there are 10 prime numbers that are less than or equal to 30.

Claim If there are k distinct prime factors appearing in the prime factorizations of all numbers in a set S, then any subset $A \subset S$ with $|A| > k + 1$, then A cannot be *good*.

Proof. By assumption, A is good with greatest common divisor c. Then, let

$$B = \left\{ \frac{a}{c} \mid a \in A \right\}.$$

Since we are dividing out by the greatest common divisor, any two numbers in B must be coprime.

Let prime p appearing as a divisor in set S. Then, note that if p appears in the factorizations of multiple numbers in B, then clearly those numbers are not coprime. Hence, it follows that p can only divide one number in B.

Since at most one number in B can be 1, and no numbers in B can be equal, we see that choosing $k + 2$ numbers would imply that at least $k + 1$ numbers of B are not equal to 1. But by pigeonhole principle, this would mean that there would have to be two numbers that are divisible by the same prime, which is contradiction. □

By our claim, Gumbo can choose at most 11 numbers. If we remove these numbers from S, we see that we are left with at least 6 prime numbers, since we can at most remove one occurrence of a number divisible by any given prime, and only 4 primes occur exactly once in our original S (namely, 17, 19, 23, 29).

It follows from our claim once more that Gabi can choose at most 7 numbers from the new S. Hence, the two can choose at most 18 distinct numbers, which we will show by example must be possible:

Gumbo can choose the numbers

$$1, 3, 5, 7, 11, 13, 16, 17, 19, 23, 29,$$

while Gabi chooses the numbers

$$2, 4, 6, 10, 14, 22, 26.$$

Thus, the answer is $30 - 18 = \boxed{12}$.

10 Via has a bag of veggie straws, which come in three colors: yellow, orange, and green. The bag contains 8 veggie straws of each color. If she eats 22 veggie straws without considering their color, what is the probability she eats all of the yellow veggie straws?

Proposed by Bradley Guo.

Answer: $\boxed{\dfrac{10}{23}}$

Solution: Let us instead consider the straws that she does not eat. She has two straws remaining at the end, if she eats all yellow straws, none of them can be yellow. There are $\binom{24}{2}$ ways to choose the color of both straws, and only $\binom{16}{2}$ for neither to be yellow, so the answer is $\frac{10}{23}$

11 We call a string of letters *purple* if it is in the form $CVCCCV$, where Cs are placeholders for (not necessarily distinct) consonants and Vs are placeholders for (not necessarily distinct) vowels. If n is the number of *purple* strings, what is the remainder when n is divided by 35? The letter y is counted as a vowel.

Proposed by Elina Lee and Nathan Cho.

Answer: $\boxed{15}$

Solution: Since y is a vowel, we know there are 6 vowels and 20 consonants. We can also see that

$$n = 20 \cdot 6 \cdot 20 \cdot 20 \cdot 20 \cdot 6,$$

and therefore, the remainder when n is divided by 5 must be zero.

Now, we look at the remainder when it's divided by 7. Note that when 20 is divided by 7, the remainder is 6. Therefore, this is the same as finding the remainder when 6^6 is divided by 7. However, $6^2 = 36$, which leaves a remainder of 1 when divided by 7. Therefore, the remainder when $6^6 = (6^2)^3$ is divided by 7 must be 1.

The number that leaves remainder of 1 when divided by 7 and 0 when divided by 5 less than 35 is 15.

12 Let a, b, c, and d be integers such that $a+b+c+d = 0$ and $(a+b)(c+d)(ab+cd) = 28$. Find $abcd$.

Proposed by Bradley Guo.

Answer: $\boxed{-8}$

Solution: Since $a+b+c+d = 0$, $(a+b) = -(c+d)$. If we let $a+b = x$, the expression can be written as $-x^2(ab+cd) = 28$. 28 is only divisible by the perfect squares 1 and 4. Therefore, x must be ± 1 or ± 2. However, due to the symmetry between a and b, and c and d, we can assume that a + b is always the positive value.

First, consider the case where $x = 1$. We know that $a + b = 1$, $c + d = -1$, and $ab + cd = -28$. We can quickly list out the possibilities of $a \cdot b$ and $c \cdot d$ and find that $ab + cd$ can not equal 28.

Next, consider the case where $x = 2$. We know that $a + b = 2$, $c + d = -2$, and $ab + cd = -7$. Similarly, we can list out the possibilities of $a \cdot b$ and $c \cdot d$ and find that a solution only occurs when $a \cdot b$ = -8, and $c \cdot d = 1$. Therefore, $abcd = -8$.

13 Griffith is playing cards. A 13-card hand with Aces of all 4 suits is known as a *godhand*. If Griffith and 3 other players are dealt 13-card hands from a standard 52-card deck, then the probability that Griffith is dealt a *godhand* can be expressed in simplest form as $\frac{a}{b}$. Find a.

Proposed by Bradley Guo.

Answer: $\boxed{\frac{11}{4165}}$

Solution: If Griffith gets a god hand, he must get all 4 aces. This means he has only nine undetermined cards out of the 48 remaining cards, so there are $\binom{48}{9}$ possible unordered hands for Griffith. On the other hand, there were a total of $\binom{52}{13}$ hands avaliable for Griffith originally, so the answer is $\frac{\binom{48}{9}}{\binom{52}{13}} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{11}{4165}$

14 For some positive integer m, the quadratic $x^2 + 202200x + 2022m$ has two (not necessarily distinct) integer roots. How many possible values of m are there?

Proposed by Joshua Hsieh.

Answer: $\boxed{50}$

Solution: We claim that $2022 \mid m$. One of the roots must have a factor of 2 becuase the product of the roots is divisible by 2. Since the sum of the roots is also divisible by 2, the other root must have a factor of 2 as well. Using similar logic for 3 and 337 proves the claim.

Thus we can write m as $2022k$. k must be the product of two numbers summing to 100. These can be $(1, 99), (2, 98), \ldots, (50, 50)$, giving an answer of 50

15 Triangle ABC with altitudes of length 5, 6, and 7 is similar to triangle DEF. If $\triangle DEF$ has integer side lengths, find the least possible value of its perimeter.

Proposed by Kevin Wu.

Answer: $\boxed{107}$

Solution: Let us denote the area of $\triangle DEF$ $\frac{A}{2}$. Since the altitudes have ratio $5 : 6 : 7$, the sides must have lengths $\frac{A}{5}, \frac{A}{6}, \frac{A}{7}$. As these are all integers, $\frac{A}{2}$ must be a multiple of $5, 6$, and 7, meaning the least possible value of $\frac{A}{2}$ is 210, and the smallest perimeter of $\triangle DEF$ is $42 + 35 + 30 = 107$.