1 What is the largest integer less than 100 that is not divisible by 2, 3, or 5?

Proposed by Bradley Guo.

Answer: $\boxed{97}$

Solution: We see that $99, 98$ are divisible by $3, 2$ respectively, but 97 works since it is prime.

2 Find the largest three digit integer which has an odd sum of digits, and an even product of digits.

Proposed by Bradley Guo.

Answer: $\boxed{988}$

Solution: Because the sum of digits is odd, then an odd number of digits are odd. Because the product is even, at least one digit is even, so at most two digits are odd. Thus, the number of odd digits is exactly 1. Since the largest digit 9 is odd and we want the largest such three digit number, the first digit must be 9 and the last two both 8.

3 How many zeros does $5! + 10! + 15! + 20! + 25!$ end in? Recall that $n! = 1 \cdot 2 \cdot \cdots \cdot n$.

Proposed by Bradley Guo.

Answer: $\boxed{1}$

Solution: First, we can notice that $5! = 120$ is divisible by 10 exactly once. Also, $10!, 15!, 20!, 25!$ are all divisible by 100 since in the product $1 \cdot 2 \cdot \cdots \cdot n$ we have both a $10, 5$, and a 2, so the last two digits are both 0. Thus, when we add them together, the last two digits will just be 20, so there is only 1 zero at the end of the sum.

4 Suppose a, b, and c are equal to $2, 3$ and 4, in some order. What's the last digit of the greatest possible value of a^{b^c}?

Proposed by Bradley Guo.

Answer: $\boxed{2}$

Solution: We claim the greatest value is 2^{3^4}. To see this, we can split into cases based on what a is. If a is 2, then $3^4 = 81 > 4^3 = 64$, so then the maximum in that case is 2^{81}. If a is 3, then $2^4 = 4^2 = 16$, so we can just consider 3^{16}. If a is 4, then $3^2 = 9 > 2^3 = 8$, so we only consider 4^9. Finally, we can see that $2^{81} > 4^9 = 2^{18}$, and that $2^{81} = 8^{27} > 3^{16}$, so in the end 2^{81} is the largest.

To compute the units digit of 2^{81}, we can note that the sequence of units digits of $2, 2^2, 2^3, \ldots$ is $2, 4, 8, 6, 2, 4, 8, 6, \ldots$, so continuing the pattern we have 2^{81} must have units digit 2.

5 Let S be the set of all even integers greater than or equal to 2022. What's the unique element n of S such that the number of divisors of $512n$ that aren't divisors of 512 is minimized?

Proposed by Kelin Zhu.

Answer: $\boxed{2048}$

Solution: We can notice that any divisor of 512 divides $512n$, so the number we're trying to minimize is $d(512n) - d(512)$, where $d(n)$ is the number of divisors of n. Since $d(512)$ is fixed, it suffices to minimize the number of divisors of $512n$. Let the prime factorization of $\frac{n}{2}$ be $2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdots$, where we know all the e_i are positive since n is even. The number of divisors of $512n$ is $(e_1 + 11)(e_2 + 1)(e_3 + 1) \cdots$. When $n = 2048$, then we get 21, but it is clear that if n has any odd prime divisors then we would get at least $11 \cdot 2 = 22$, so 2048 minimizes $d(512n)$ as desired.

6 In a regular 10 by 10 multiplication table, the numbers that would appear are the products ab for every a ranging from 1 to 10 and every b ranging from 1 to 10.

A wrong multiplication table is a multiplication table that only keeps the last digit of the product instead of entire product. In a 10 by 10 wrong multiplication table starting from 1, what is the least number of times that any result appears?

Proposed by Albert Wang.

Answer: $\boxed{4}$

Solution: By the Chinese remainder theorem, we can analyze the probability to get a number congruent to n mod 2 and mod 5 separately, and then multiply them to get the probability of getting n. Since we want the minimum number of times a result appears, we can multiply the minimum probabilities. The probability of getting an odd number is $\frac{1}{4}$ since it requires both a, b to be odd. Looking mod 5, the probability of getting 0 is $\frac{9}{25}$, and the probability of any other remainder is $\frac{4}{25}$, so the minimum overall probability is $\frac{1}{4} \cdot \frac{4}{25} = \frac{1}{25}$, so the smallest number is $\frac{100}{25} = 4$.

7 Find the number of ordered pairs of positive integers (a, b) such that the least common multiple of a and b is $13^{29} \cdot 29^{13}$.

Proposed by Nathan Cho.

Answer: $\boxed{1593}$

Solution: We see that a, b can be written as $13^e \cdot 29^f, 13^g \cdot 29^h$, since any prime divisor of either a, b would have to divide the least common multiple. Then, we know that the least common multiple will be $13^{\max(e,g)} \cdot 19^{\max(f,h)}$, so we need both $e, g \leq 13, f, h \leq 29$ and one of $e, g = 13$ and one of $f, h = 29$. There are $2 \cdot 13 + 1 = 27$ pairs for e, g and $2 \cdot 29 + 1 = 59$ pairs for f, h giving $27 \cdot 59 = 1593$ possiblities total.

8 Two items have prices \$$a.bc$ and \$$d.ef$ for digits a, b, c, d, e, f. When the cashier finds their value, he gets the same result regardless if he added them or multiplied them. Find the largest possible value of the digit d.

Proposed by Albert Wang.

Answer: $\boxed{7}$

Solution: Letting $a.bc = x, d.ef = y$, then we need $x + y = xy$. We can rearrange this as $xy - x - y + 1 = 1$, so $(x-1)(y-1) = 1$. Multiplying by 100, we get $(abc-100)(def-100) = 10000$. Since abc, def are three digit integers, we know that $def - 100$ must be a divisor of 10000. We can see the largest divisor of 10000 is 625, so the largest value of d is 7, with $a.bc = 1.16, d.ef = 7.25$.