1 What is $1 + 2 \cdot 3$?

Proposed by Bradley Guo.

Answer: $\boxed{7}$

Solution: We simply compute: $2 \cdot 3 = 6$, $6 + 1 = 7$

2 A square of side length 2 is cut into 4 congruent squares. What is the perimeter of one of the 4 squares?

Proposed by Bradley Guo.

Answer: $\boxed{4}$

Solution: Each small square has one fourth the area of the original square, and therefore one half the sidelength. Squares have four equal sides, so the desired perimeter is $4 \cdot \frac{2}{2} = 4$.

3 6 people split a bag of cookies such that they each get 21 cookies. Kyle comes and demands his share of cookies. If the 7 people then re-split the cookies equally, how many cookies does Kyle get?

Proposed by Kevin Yao.

Answer: $\boxed{18}$

Solution: From the first splitting, we see that there are $6 \cdot 21 = 126$ cookies in total. To find how many cookies Kyle gets, we divide this by 7 to get $\frac{126}{7} = 18$.

4 Blobby flips 4 coins. What is the probability he sees at least one heads and one tails?

Proposed by Nathan Cho.

Answer: $\boxed{\frac{7}{8}}$

Solution: The only way for Blobby to utterly fail in seeing both at least one head and one tail is to only see either heads or tails. The probability he only sees heads is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, as is the probability he only sees tails, so the probability he doesn't see at least one heads and one tails is $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. The probability he does see them is $1 - \frac{1}{8} = \frac{7}{8}$

5 The product of 10 consecutive positive integers ends in 3 zeros. What is the minimum possible value of the smallest of the 10 integers?

Proposed by Bradley Guo.

Answer: $\boxed{16}$

Solution: To make a trailing zero, we need a factor of 2 and a factor of 5. There's always going to be more 2s than 5s, so we focus on how many 5s there are. These come from the multiples of 5 in that set of 10 integers. However, we can't fit three multiples of 5 into that small a range, so one multiple has to pull extra weight and be divisible by 25. The smallest multiple of 25 is 25, and the smallest starting number that will contain 25 is $25 - 10 + 1 = 16$.

6 How many perfect squares less than 100 are odd?

Proposed by Bradley Guo.

Answer: $\boxed{5}$

Solution: The largest odd square under 100 is $81 = 9^2$. As odd squares can only be obtained by squaring odd numbers, we only need to count the number of odd numbers less than or equal to 9, which is 5.

7 Guuce continually rolls a fair 6-sided dice until he rolls a 1 or a 6. He wins if he rolls a 6, and loses if he rolls a 1. What is the probability that Guuce wins?

Proposed by Bradley Guo.

Answer: $\boxed{\dfrac{1}{2}}$

Solution: Guuce's two ending states are equally probable: there is no reason to favor landing on a 6 over landing on a 1. We also know that Guuce must either win or lose, so the probability of winning and the probability of losing must sum to 1 The probability that Guuce wins is then $\frac{1}{2}$.

8 The perimeter and area of a square with integer side lengths are both three digit integers. How many possible values are there for the side length of the square?

Proposed by Bradley Guo.

Answer: $\boxed{7}$

Solution: If the square has sidelength s, the perimeter is $4s$, and the area is s^2. $4s$ is a three digit integer when $s \geq \frac{100}{4} = 25$ and $s < \frac{1000}{4} = 250$, and s^2 is a three digit integer when $s \geq \sqrt{100} = 10$ and $s \leq \sqrt{961} = 31$. Taking the highest lower bound and the lowest upper bound to find the range where both are satisfied, we find $25 \leq s \leq 31$. There are $31 - 25 + 1 = 7$ integers in this range, so that is our answer.

9 In the coordinate plane, a point is selected in the rectangle defined by $-6 \leq x \leq 4$ and $-2 \leq y \leq 8$. What is the largest possible distance between the point and the origin, $(0, 0)$?

Proposed by Justin Chen.

Answer: $\boxed{10}$

Solution: This point has to be a corner of the rectangle, since if it was inside the rectangle or on one side we could always move it in the direction some side of the rectangle and get farther away from the origin. The farthest corner of this kind of rectangle on the plane has the coordinates with the largest absolute magnitudes, which is $(-6, 8)$. This point is $\sqrt{(-6)^2 + 8^2} = 10$ away from the origin, so that is our answer.

10 The sum of two numbers is 6 and the sum of their squares is 32. Find the product of the two numbers.

Proposed by Bradley Guo.

Answer: $\boxed{2}$

Solution: If the numbers are a and b, we are told $a+b=6$, $a^2+b^2=32$, and that we want to find ab. We note that $(a+b)^2=36$, so expanding reveals $a^2+2ab+b^2=36$. We see the ab sitting right there, so we subtract $a^2+b^2=32$ from both sides and divide by 2 to find $ab=\frac{4}{2}=2$.

11 How many two digit numbers are there such that the product of their digits is prime?

Proposed by Bradley Guo.

Answer: $\boxed{8}$

Solution: We note that multiplying together two integers that are both not 1 cannot equal a prime, since those integers would be factors of the product. Therefore, one of the two digits must be 1, and the other must equal the product divided by 1, which is just some 1 digit prime. There are 4 such primes $(2,3,5,7)$ and 2 ways to place the 1 in the number, so our answer is $2\cdot 4=8$

12 Triangle ABC has area 4 and $\overline{AB}=4$. What is the maximum possible value of $\angle ACB$?

Proposed by Bradley Guo.

Answer: $\boxed{90^\circ}$

Solution: We know that the height of ABC that goes to side AB is $\frac{4\cdot 2}{4}=2$, so point C is locked to be 2 away from the line A and B are on. Sliding C around on this line (drawing some pictures), we can see that $\angle ACB$ is maximized when C is right in between A and B such that $\overline{AC}=\overline{BC}$. This is an isoceles triangle with base 4 and height 2. The two legs therefore have length $\sqrt{2^2+\left(\frac{4}{2}\right)^2}=2\sqrt{2}$. This triangle is revealed to be a right isoceles triangle with base AB, so $\angle ACB$ is maximized at 90°

13 Let $ABCD$ be an iscoceles trapezoid with $AB=CD$ and M be the midpoint of $\overline{AD}$. If $\triangle ABM$ and $\triangle MCD$ are equilateral, and $BC=4$, find the area of trapezoid $ABCD$.

Proposed by Bradley Guo.

Answer: $\boxed{12\sqrt{3}}$

Solution: As $AB=CD$, $\triangle ABM$ and $\triangle MCD$ have the same sidelength, so $BM=CM$. We also note that since $\angle AMB+\angle DMC+\angle BMC=180^\circ$ and $\angle AMB=\angle DMC=60^\circ$, $\angle BMC$ is also 60°. Combined, these two facts tell us that $\triangle BMC$ is also equilateral with the same sidelength as the other two triangles, so $ABCD$ is just three equilateral triangles put together. As $BC=4$, all three have sidelength 4, so the entire trapezoid has area $3\cdot\frac{4^2\sqrt{3}}{4}=12\sqrt{3}$.

14 Let x and y be positive real numbers that satisfy $(x^2 + y^2)^2 = y^2$. Find the maximum possible value of x.

Proposed by Bradley Guo.

Answer: $\boxed{\frac{1}{2}}$

Solution: Taking the square root of both sides of the equation, we find that $x^2 + y^2 = y$. It follows that

$$x = \sqrt{y(1-y)}$$

which is maximized at $y = 0.5$, at which x is also 0.5.

15 Let AOB be a quarter circle with center O and radius 4. Let ω_1 and ω_2 be semicircles inside AOB with diameters OA and OB, respectively. Find the area of the region within AOB but outside of ω_1 and ω_2.

Proposed by Justin Chen.

Answer: $\boxed{2\pi - 4}$

Solution: The two semicircles have radii $\frac{4}{2} = 2$, and are a distance $\sqrt{2^2 + 2^2} = 2\sqrt{2}$ apart. Their intersection is the sum of two circular segments (circular sectors minus an inscribed triangle). O, the centers of ω_1 and ω_2, and the intersection of ω_1 and ω_2 not at O form a square, so these segments span 90° arcs. Each one then has an area of $\frac{90°}{360°} \cdot 2^2\pi - \frac{2 \cdot 2}{2}$, so the total intersection is $2\pi - 4$. The union of the two semicircles is then $\frac{2^2\pi}{2} \cdot 2 - 2\pi + 4 = 2\pi + 4$, so the area in the quarter circle but not in a semicircle is $\frac{4^2\pi}{4} - 2\pi - 4 = 2\pi - 4$.

16 Positive integers a, b, c form a geometric sequence with an integer common ratio. If $c = a + 56$, find b.

Proposed by Bradley Guo.

Answer: $\boxed{21}$

Solution: Let the common ratio in the geometric series be r. Then $ar^2 = c = a + 56$. The equation $a(r^2 - 1) = 56$ gives us $a = 7, r = 3$ as the only integer options. This means that $b = ar = 21$.

17 In parallelogram $ABCD$, $\angle A \cdot \angle C - \angle B \cdot \angle D = 720°$ where all angles are in degrees. Find the value of $\angle C$.

Proposed by Bradley Guo.

Answer: $\boxed{92}$

Solution: This is a parallelogram, so we have $\angle A = \angle C$, $\angle B = \angle D$, and $\angle B = 180° - \angle C$. Substituting these values into the given equation:

$$C^2 - (180 - C)^2 = 720$$

$$(C - (180 - C))(C + (180 - C)) = 720$$

So $C = 92$.

18 Steven likes arranging his rocks. A mountain formation is where the sequence of rocks to the left of the tallest rock increase in height while the sequence of rocks to the right of the tallest rock decrease in height. If his rocks are $1, 2, \ldots, 10$ inches in height, how many mountain formations are possible? For example: the sequences (1-3-5-6-10-9-8-7-4-2) and (1-2-3-4-5-6-7-8-9-10) are considered mountain formations.

Proposed by Christopher Jin.

Answer: $\boxed{512}$

Solution: We note that once we choose what rocks lie on the left of the tallest rock, everything is locked into place, since there is only one way to arrange those rocks and the rocks on the other side are effectively chosen and ordered in one way as well. This means that the number of mountain formations is simply the number of subsets of $1, 2, 3 \ldots 9$ you can put on the left of 10, which is just $2^9 = 512$, since each rock can either be on the left or not.

19 Find the smallest 5-digit multiple of 11 whose sum of digits is 15.

Proposed by Bradley Guo.

Answer: $\boxed{10329}$

Solution: If a number is a multiple of 11, then the sum of the odd indexed digits (ones place, hundreds place, ten thousands place) and the sum of the even indexed digits (tens place, thousands place) differ by a multiple of 11. Here, the sum of all the digits is 15, meaning this difference can't be 0 or anything more than 11. Therefore, the two sums are 2 and $2 + 11 = 13$. We want to minimize the number, so we drop the ten thousands place to 1 and the thousands place to 2. The tens place must be $2 - 0 = 2$ while the ones place and hundreds place must sum to 12, so we minimize everything with 10329 by forcing the hundreds digit to be as small as possible.

20 Two circles, ω_1 and ω_2, have radii of 2 and 8, respectively, and are externally tangent at point P. Line l is tangent to the two circles, intersecting ω_1 at A and ω_2 at B. Line m passes through P and is tangent to both circles. If line m intersects line l at point Q, calculate the length of PQ.

Proposed by Yunyi Ling.

Answer: $\boxed{4}$

Solution: By symmetry, we see that $QP = QA$, $QP = QB$, so then $QP = \frac{QA+QB}{2} = \frac{AB}{2}$. Let O_1, O_2 be the centers of ω_1, ω_2, and construct C such that $ABCO_1$ is a rectangle. Then, we know that $CB = 2$, so $CO_2 = 6$. By Pythagorean theorem on triangle $\triangle CO_1O_2$, we find that $CO_1 = \sqrt{10^2 - 6^2} = 8$, so then $AB = 8$ and $QP = 4$.

21 Sen picks a random 1 million digit integer. Each digit of the integer is placed into a list. The probability that the last digit of the integer is strictly greater than twice the median of the digit list is closest to $\frac{1}{a}$, for some integer a. What is a?

Proposed by Bradley Guo.

Answer: $\boxed{20}$

Solution: Since we have so many digits, then we know that the probability of the median being 4 or 5 are both very close to $\frac{1}{2}$. If the median is 5, then there is 0 probability we get a digit larger than 10, but if the median is 4 there is a $\frac{1}{10}$ probability we get a number larger than 8, so we get $\frac{1}{10} \cdot \frac{1}{2} = \frac{1}{20}$.

22 Let 6 points be evenly spaced on a circle with center O, and let S be a set of 7 points: the 6 points on the circle and O. How many equilateral polygons (not self-intersecting and not necessarily convex) can be formed using some subset of S as vertices?

Proposed by Elina Lee.

Answer: $\boxed{27}$

Solution: First, we count the number of equilateral triangles. There are 6 with O as a vertex and 2 without. Next, we count the number of rhombi. There are 6 of these. Next, we count the number of pentagons. There are none. For hexagons, we can include the whole hexagon, or we can include the hexagon with a rhombus missing. This gives 7. For heptagons, we can only have the whole hexagon without an equilateral triangle, this gives 6.

Adding gives 27.

23 For a positive integer n, define r_n recursively as follows: $r_n = r_{n-1}^2 + r_{n-2}^2 + \ldots + r_0^2$, where $r_0 = 1$. Find the greatest integer less than

$$\frac{r_2}{r_1^2} + \frac{r_3}{r_2^2} + \ldots + \frac{r_{2023}}{r_{2022}^2}.$$

Proposed by Kelin Zhu.

Answer: $\boxed{2023}$

Solution: We can see that $r_n = r_{n-1} + r_{n-1}^2 = r_{n-1}(r_{n-1} + 1)$, so $\frac{r_n}{r_{n-1}^2} = 1 + \frac{1}{r_{n-1}}$. We compute that $r_2 = 2$, and then since $r_n > r_{n-1}^2$, we have that $r_i > 2^{i-1}$. Therefore,

$$\sum_{i=1}^{2022} \frac{r_{i+1}}{r_i^2} = 2022 + \sum_{i=1}^{2022} \frac{1}{r_i} < 2022 + \sum_{i=1}^{2022} \frac{1}{2^{i-1}} < 2024.$$

On the other hand, we know that $\frac{1}{r_1} = 1$, so the sum is clearly at least 2023 and we get that the greatest integer less than this sum is 2023.

24 Arnav starts at 21 on the number line. Every minute, if he was at n, he randomly teleports to $2n^2, n^2$, or $\frac{n^2}{4}$ with equal chance. What is the probability that Arnav only ever steps on integers?

Proposed by Kelin Zhu.

Answer: $\boxed{\dfrac{2}{5}}$

Solution: Arnav can be in 3 possible states. He can be on an odd integer, a even non-multiple of 4, or a multiple of 4. If Arnav is on a multiple of 4, he cannot ever step on a non-integer because even if he always teleports to $\frac{n^2}{4}$, the number is he on will remain a multiple of 4. Let the probability that Arnav wins starting from an odd integer be x. Starting from an odd integer, he must teleport to $2n^2$ before $\frac{n^2}{4}$, which has chance $\frac{1}{2}$. After doing so, he is on a even non-multiple of 4. If he teleports to $2n^2$ or n^2, he will be on a multiple of 4, and therefore win immediately. If he teleports to $\frac{n^2}{4}$, he will be back on an odd integer, and have probability x to win. Thus, we have the equation,

$$x = \frac{1}{2}(\frac{2}{3} + \frac{1}{3}x)$$

which gives $x = \frac{2}{5}$.

25 Let $ABCD$ be a rectangle inscribed in circle ω with $AB = 10$. If P is the intersection of the tangents to ω at C and D, what is the minimum distance from P to AB?

Proposed by Nathan Cho.

Answer: $\boxed{10\sqrt{2}}$

Solution: Let O be the center of circle ω, the intersection of PO with CD be E, and BC be $2x$. Since $\angle OCP = 90$, we know that

$$\triangle OCP \sim \triangle OEC$$

Also, $OE = x$ and $EC = 5$, so by the Pythagorean Theorem, $OC = \sqrt{x^2 + 25}$. From similar triangles $\triangle OCP$ and $\triangle OEC$, we have

$$\frac{OE}{OC} = \frac{OC}{OP}$$

$$OP = \frac{OC^2}{OE} = \frac{x^2 + 25}{x}$$

Since the distance from P to AP is just $OP + x$, we want to maximize $\frac{x^2+25}{x} + x = 2x + \frac{25}{x}$. Using the AM-GM inequality,

$$\frac{2x + \frac{25}{x}}{2} \geq \sqrt{2x \cdot \frac{25}{x}} = 5\sqrt{2}$$

Thus, the minimum possible value of $2x + \frac{25}{x}$ is $10\sqrt{2}$.

26 For every person who wrote a problem that appeared on the final MBMT tests, take the number of problems they wrote, and then take that number's factorial, and finally multiply all these together to get n. Estimate the greatest integer a such that 2^a evenly divides n.

Proposed by Nathan Cho.

Answer: $\boxed{81}$

Solution: N/A

27 Circles of radius 5 are centered at each corner of a square with side length 6. If a random point P is chosen randomly inside the square, what is the probability that P lies within all four circles?

Proposed by Bradley Guo.

Answer: $\boxed{0.0608251516782331189}$

Solution: Let the square be $ABCD$, the midpoint of AB be M, the center of the square be O, and the intersection between the circle centered at A and B be Q. Since $AQ = 5$ and $AM = 3$, we know $QM = 4$ and therefore $OQ = 1$. At this point, we can approximate the area of the region of interest as a square with diagonal length 2, which gives an answer of $\frac{2}{18} \approx 0.0556$. The true answer is approximately $\frac{2.1897}{36} \approx 0.0608$

28 Mr. Rose's evil cousin, Mr. Caulem, has teaches a class of three hundred bees. Every week, he tries to disrupt Mr. Rose's 4th period by sending three of his bee students to fly around and make human students panic. Unfortunately, no pair of bees can fly together twice, as then Mr. Rose will become suspicious and trace them back to Mr. Caulem. What's the largest number of weeks Mr. Caulem can disrupt Mr. Rose's class?

Proposed by Nathan Cho.

Answer: $\boxed{14900}$

Solution: N/A

29 Two blind brothers Beard and Bored are driving their tractors in the middle of a field facing north, and both are 10 meters west from a roast turkey. Beard, can turn exactly $0.7°$ and Bored can turn exactly $0.2°$ degrees. Driving at a consistent 2 meters per second, they drive straight until they notice the smell of the turkey getting farther away, and then turn right and repeat until they get to the turkey.

Suppose Beard gets to the Turkey in about 818.5 seconds. Estimate the amount of time it will take Bored.

Proposed by Nathan Cho.

Answer: $\boxed{2864.786}$

Solution: Both brothers basically spiral in towards the center. If they are currently x meters from the turkey, and they can turn at a θ angle, then they will end up $x \cos\theta$

meters away from the turkey and travel $x\sin\theta$ meters. Therefore, in the end, the brother will travel

$$10\sin\theta + 10\cos\theta\sin\theta + 10\cos^2\theta\sin\theta + ... = \frac{10\sin\theta}{1-\cos\theta} = 10\cot\theta/2,$$

and dividing by the speed will yield the amount of time it will take. If we could just plug in $\theta = 0.2°$, we would be done. In fact, this yields about 2864.786 after dividing by 2 to account for the speed, when plugged into a calculator.

We are given that

$$5\cot 0.35° \approx 818.5.$$

Since $y = x$ approximates $\tan x$ for values of x close to zero, we can say that dividing the angle by 10 when it is close to zero will approximately multiply the entire thing by 10. Therefore, we can reasonably deduce that

$$5\cot 0.1° = \frac{5}{\tan(0.1°)} \approx 3.5\frac{5}{\tan(0.35°)} = 0.35 \cdot 5\cot 0.35° \approx 2864.75.$$

Noting that $\tan x > x$ for the x we are concerned with tells us that our fraction increases by more than a factor of 3.5, but very close to a factor of 3.5. By how much more, however, is a harder question to answer.

30 Let a be the probability that 4 randomly chosen positive integers have no common divisor except for 1. Estimate $300a$. Note that the integers $1, 2, 3, 4$ have no common divisor except for 1.

Remark. This problem is asking you to find

$$300\lim_{n\to\infty} a_n,$$

if a_n is defined to be the probability that 4 randomly chosen integers from $\{1, 2, ..., n\}$ have greatest common divisor 1.

Proposed by Nathan Cho.

Answer: $\boxed{\frac{27000}{\pi^4} \approx 277.181520876...}$

Solution: For any prime number p, we must have that p does not divide any of the numbers. The probability that any number is divisible by p is $1/p$. Hence, the probability that there exists one number not divisible by p is $1 - p^{-4}$.

From here, there are two ways to finish, one of which requires advanced background knowledge, and another that reduces the problem through approximations which gives you full points.

Approximating for full points

We note that when p is big, then p^{-4} is very small. Namely, note that $p > 10$ implies that $p^{-4} < 0.01\%$. It is up to the competitor how much computation they want to do, for the number of points they receive. Surprisingly, only 2 terms are needed to get within 1 of the answer:

$$\left(1 - 2^{-4}\right)\left(1 - 3^{-4}\right) 300 = 277.\overline{7},$$

which is enough for full points.

Closed form, with high powered theory

Earlier, we found that our answer is

$$\prod_p \left(1 - p^{-4}\right)$$

$$\left(\prod_p \frac{1}{1 - p^{-4}}\right)^{-1}$$

We note that this is an Euler product, namely, this becomes

$$\left(\prod_p \frac{1}{1 - p^{-4}}\right)^{-1}$$

$$= \prod_p \left(1 - p^{-4}\right)$$

$$= \left(\prod_p \sum_{k=0}^{\infty} \frac{1}{p^{4k}}\right)^{-1}$$

$$= \left(\left(\frac{1}{1} + \frac{1}{2^4} + \frac{1}{2^8} + \cdots\right)\left(\frac{1}{1} + \frac{1}{3^4} + \frac{1}{3^8} + \cdots\right)\left(\frac{1}{1} + \frac{1}{5^4} + \cdots\right)\cdots\right)^{-1}$$

$$= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)^{-1}$$

$$= \left(\sum_{k=1}^{\infty} \frac{1}{k^4}\right)^{-1}$$

$$= \frac{1}{\zeta(4)} = \frac{90}{\pi^4}$$

Hence, the answer is

$$\frac{27000}{\pi^4} \approx \boxed{277.181520876...}$$