1 What is $1 + 2 \cdot 3$?

Proposed by Bradley Guo.

Answer: $\boxed{7}$

Solution: We simply compute: $2 \cdot 3 = 6$, $6 + 1 = 7$

2 What is the average of the first 9 positive integers?

Proposed by Bradley Guo.

Answer: $\boxed{5}$

Solution: we take the sum of the first 9 positive integers, and divide by 5. The sum is given by $\frac{9(9+1)}{2} = 45$, so our answer is $\frac{45}{9} = \boxed{5}$

3 A square of side length 2 is cut into 4 congruent squares. What is the perimeter of one of the 4 squares?

Proposed by Bradley Guo.

Answer: $\boxed{4}$

Solution: Each small square has one fourth the area of the original square, and therefore one half the sidelength. Squares have four equal sides, so the desired perimeter is $4 \cdot \frac{2}{2} = 4$.

4 Find the ratio of a circle's circumference squared to the area of the circle.

Proposed by Bradley Guo.

Answer: $\boxed{4\pi}$

Solution: If the circle has radius r, its circumference is $2\pi r$, and its area is πr^2. Squaring the circumference and dividing by the area gives $\frac{4\pi^2 r^2}{\pi r^2} = \boxed{4\pi}$.

5 6 people split a bag of cookies such that they each get 21 cookies. Kyle comes and demands his share of cookies. If the 7 people then re-split the cookies equally, how many cookies does Kyle get?

Proposed by Kevin Yao.

Answer: $\boxed{18}$

Solution: From the first splitting, we see that there are $6 \cdot 21 = 126$ cookies in total. To find how many cookies Kyle gets, we divide this by 7 to get $\frac{126}{7} = 18$.

6 How many prime numbers are perfect squares?

Proposed by Bradley Guo.

Answer: $\boxed{0}$

Solution: If a number is a perfect square, it can be written as n^2 for some positive n. n is a divisor of this number, so unless n also is n^2, n^2 has 3 divisors and is not prime. If $n = n^2$, $n = 1$ and is by definition not prime.

7 Josh has an unfair 4-sided die numbered 1 through 4. The probability it lands on an even number is twice the probability it lands on an odd number. What is the probability it lands on either 1 or 3?

Proposed by Ivy Guo.

Answer: $\boxed{\dfrac{1}{3}}$

Solution: The probability of rolling an odd or an even is 1, so if the probability of rolling odd is p, then $p + 2p = 1$, so $p = \frac{1}{3}$.

8 If Alice consumes 1000 calories every day and burns 500 every night, how many days will it take for her to first reach a net gain of 5000 calories?

Proposed by Steven Wang.

Answer: $\boxed{9}$

Solution: Each day and night cycle, Alice gains a net $1000 - 500 = 500$ calories, so after the 8th night, she gains $500 \cdot 8 = 4000$ calories. On the 9th day, she gains 1000 calories reaching a net gain of $4000 + 1000 = 5000$ calories for the first time.

9 Blobby flips 4 coins. What is the probability he sees at least one heads and one tails?

Proposed by Nathan Cho.

Answer: $\boxed{\dfrac{7}{8}}$

Solution: The only way for Blobby to utterly fail in seeing both at least one head and one tail is to only see either heads or tails. The probability he only sees heads is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, as is the probability he only sees tails, so the probability he doesn't see at least one heads and one tails is $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. The probability he does see them is $1 - \frac{1}{8} = \frac{7}{8}$

10 Lillian has n jars and 48 marbles. If George steals one jar from Lillian, she can fill each jar with 8 marbles. If George steals 3 jars, Lillian can fill each jar to maximum capacity. How many marbles can each jar fill?

Proposed by Tong-Tong Ye.

Answer: $\boxed{12}$

Solution: When George steals one jar, Lilian is left with $n-1$ jars, so the total number of marbles in those jars is $8(n-1)$. This equals 48, so $n-1 = \frac{48}{8} = 6$ and $n = 7$. When George instead steals 3 jars, Lilian is left with only 4, and each jar is filled with $\frac{48}{4} = 12$ marbles. We are told that this is their maximum capacity, so we are done.

11 How many perfect squares less than 100 are odd?

Proposed by Bradley Guo.

Answer: $\boxed{5}$

Solution: The largest odd square under 100 is $81 = 9^2$. As odd squares can only be obtained by squaring odd numbers, we only need to count the number of odd numbers less than or equal to 9, which is 5.

12 Jash and Nash wash cars for cash. Jash gets \$6 for each car, while Nash gets \$11 per car. If Nash has earned \$1 more than Jash, what is the least amount of money that Nash could have earned?

Proposed by Nathan Shan.

Answer: $\boxed{55}$

Solution: Let N be the number of cars Nash has washed, and let J be the number of cars Jash has washed. We are given that $11N = 6J + 1$, so $11N \equiv 5N \equiv -N \equiv 1 \mod 6$. This means that $N \equiv -1 \equiv 5 \mod 6$, so the smallest value of N is 5. The smallest amount of money Nash could have earned is then $11N = 55$.

13 The product of 10 consecutive positive integers ends in 3 zeros. What is the minimum possible value of the smallest of the 10 integers?

Proposed by Bradley Guo.

Answer: $\boxed{16}$

Solution: To make a trailing zero, we need a factor of 2 and a factor of 5. There's always going to be more 2s than 5s, so we focus on how many 5s there are. These come from the multiples of 5 in that set of 10 integers. However, we can't fit three multiples of 5 into that small a range, so one multiple has to pull extra weight and be divisible by 25. The smallest multiple of 25 is 25, and the smallest starting number that will contain 25 is $25 - 10 + 1 = 16$.

14 Guuce continually rolls a fair 6-sided dice until he rolls a 1 or a 6. He wins if he rolls a 6, and loses if he rolls a 1. What is the probability that Guuce wins?

Proposed by Bradley Guo.

Answer: $\boxed{\frac{1}{2}}$

Solution: Guuce's two ending states are equally probable: there is no reason to favor landing on a 6 over landing on a 1. We also know that Guuce must either win or lose, so the probability of winning and the probability of losing must sum to 1 The probability that Guuce wins is then $\frac{1}{2}$.

15 The perimeter and area of a square with integer side lengths are both three digit integers. How many possible values are there for the side length of the square?

Proposed by Bradley Guo.

Answer: $\boxed{7}$

Solution: If the square has sidelength s, the perimeter is $4s$, and the area is s^2. $4s$ is a three digit integer when $s \geq \frac{100}{4} = 25$ and $s < \frac{1000}{4} = 250$, and s^2 is a three digit integer when $s \geq \sqrt{100} = 10$ and $s \leq \sqrt{961} = 31$. Taking the highest lower bound and the lowest upper bound to find the range where both are satisfied, we find $25 \leq s \leq 31$. There are $31 - 25 + 1 = 7$ integers in this range, so that is our answer.

16 The cooking club at Blair creates 14 croissants and 21 danishes. Daniel chooses pastries randomly, stopping when he gets at least one croissant and at least two danishes. How many pastries must he choose to guarantee that he has one croissant and two danishes?

Proposed by Jason Youm.

Answer: $\boxed{22}$

Solution: We note that if Daniel blindly picks 21 or less pastries, he could have gotten all danishes. However, if he gets one more, he knows that even in the case where he got the absolute least number of croissants, he still got $22 - 21 = 1$ croissant. In the case where he got the minimal amount of danishes and took all the croissants, he still got $22 - 14 = 8 > 1$ danishes, so 22 is our answer.

17 Each digit in a 3 digit integer is either 1, 2, or 4 with equal probability. What is the probability that the hundreds digit is greater than the sum of the tens digit and the ones digit?

Proposed by Bradley Guo.

Answer: $\boxed{\dfrac{1}{9}}$

Solution: We note that if there is a 4 in the tens or ones place, this is impossible, since the largest value the hundreds digit can take is 4. Assuming this doesn't happen, the last two digits can sum to 2 and equal 1 and 1, sum to 3 and equal 1 and 2 in some order, or sum to 4 and equal 2 and 2. In the two cases, the hundreds digit has to be 4, and in the last case there is no value for the hundreds digit that works. This leaves us with an answer of $\frac{3}{3^3} = \frac{1}{9}$, since there are 3 numbers that work out of the 27 that exist.

18 How many two digit numbers are there such that the product of their digits is prime?

Proposed by Bradley Guo.

Answer: $\boxed{8}$

Solution: We note that multiplying together two integers that are both not 1 cannot equal a prime, since those integers would be factors of the product. Therefore, one of the two digits must be 1, and the other must equal the product divided by 1, which is just some 1 digit prime. There are 4 such primes $(2, 3, 5, 7)$ and 2 ways to place the 1 in the number, so our answer is $2 \cdot 4 = 8$

19 In the coordinate plane, a point is selected in the rectangle defined by $-6 \leq x \leq 4$ and $-2 \leq y \leq 8$. What is the largest possible distance between the point and the origin, $(0, 0)$?

Proposed by Justin Chen.

Answer: $\boxed{10}$

Solution: This point has to be a corner of the rectangle, since if it was inside the rectangle or on one side we could always move it in the direction some side of the rectangle and get farther away from the origin. The farthest corner of this kind of rectangle on the plane has the coordinates with the largest absolute magnitudes, which is $(-6, 8)$. This point is $\sqrt{(-6)^2 + 8^2} = 10$ away from the origin, so that is our answer.

20 The sum of two numbers is 6 and the sum of their squares is 32. Find the product of the two numbers.

Proposed by Bradley Guo.

Answer: $\boxed{2}$

Solution: If the numbers are a and b, we are told $a + b = 6$, $a^2 + b^2 = 32$, and that we want to find ab. We note that $(a + b)^2 = 36$, so expanding reveals $a^2 + 2ab + b^2 = 36$. We see the ab sitting right there, so we subtract $a^2 + b^2 = 32$ from both sides and divide by 2 to find $ab = \frac{4}{2} = 2$.

21 Triangle ABC has area 4 and $\overline{AB} = 4$. What is the maximum possible value of $\angle ACB$?

Proposed by Bradley Guo.

Answer: $\boxed{90^\circ}$

Solution: We know that the height of ABC that goes to side AB is $\frac{4 \cdot 2}{4} = 2$, so point C is locked to be 2 away from the line A and B are on. Sliding C around on this line (drawing some pictures), we can see that $\angle ACB$ is maximized when C is right in between A and B such that $\overline{AC} = \overline{BC}$. This is an isoceles triangle with base 4 and height 2. The two legs therefore have length $\sqrt{2^2 + \left(\frac{4}{2}\right)^2} = 2\sqrt{2}$. This triangle is revealed to be a right isoceles triangle with base AB, so $\angle ACB$ is maximized at 90°

22 Let $ABCD$ be an iscoceles trapezoid with $AB = CD$ and M be the midpoint of $\overline{AD}$. If $\triangle ABM$ and $\triangle MCD$ are equilateral, and $BC = 4$, find the area of trapezoid $ABCD$.

Proposed by Bradley Guo.

Answer: $\boxed{12\sqrt{3}}$

Solution: As $AB = CD$, $\triangle ABM$ and $\triangle MCD$ have the same sidelength, so $BM = CM$. We also note that since $\angle AMB + \angle DMC + \angle BMC = 180°$ and $\angle AMB = \angle DMC = 60°$, $\angle BMC$ is also $60°$. Combined, these two facts tell us that $\triangle BMC$ is also equilateral with the same sidelength as the other two triangles, so $ABCD$ is just three equilateral triangles put together. As $BC = 4$, all three have sidelength 4, so the entire trapezoid has area $3 \cdot \frac{4^2\sqrt{3}}{4} = 12\sqrt{3}$.

23 Let x and y be positive real numbers that satisfy $(x^2 + y^2)^2 = y^2$. Find the maximum possible value of x.

Proposed by Bradley Guo.

Answer: $\boxed{\frac{1}{2}}$

Solution: Taking the square root of both sides of the equation, we find that $x^2 + y^2 = y$. It follows that

$$x = \sqrt{y(1-y)}$$

which is maximized at $y = 0.5$, at which x is also 0.5.

24 In parallelogram $ABCD$, $\angle A \cdot \angle C - \angle B \cdot \angle D = 720°$ where all angles are in degrees. Find the value of $\angle C$.

Proposed by Bradley Guo.

Answer: $\boxed{92}$

Solution: This is a parallelogram, so we have $\angle A = \angle C$, $\angle B = \angle D$, and $\angle B = 180° - \angle C$. Substituting these values into the given equation:

$$C^2 - (180 - C)^2 = 720$$

$$(C - (180 - C))(C + (180 - C)) = 720$$

So $C = 92$.

25 The number $12ab9876543$ is divisible by 101, where a, b represent digits between 0 and 9. What is $10a + b$?

Proposed by Nathan Shan.

Answer: $\boxed{58}$

Solution: We slowly eat away at the digits of this number by subtracting relevant multiples of 101. Subtracting 303 gives $12ab9876240$, and 4040 more gives $12ab9872200$. We continue taking the last digit, multiplying it by 101, and then tacking on the trailing zeros, until we get to the top. This gives the sequence $12ab9852000$, $12ab9650000$, and $12ab4600000$. Now, we can drop the zeroes because 10 does not divide 101, to find $12ab46 = 101N = 100N + N$ for some N. The last two digits of N are 46 and the first two are 12, so $ab = 46 + 12 = 58$.

26 For every person who wrote a problem that appeared on the final MBMT tests, take the number of problems they wrote, and then take that number's factorial, and finally multiply all these together to get n. Estimate the greatest integer a such that 2^a evenly divides n.

Proposed by Nathan Cho.

Answer: $\boxed{81}$

Solution: N/A

27 Circles of radius 5 are centered at each corner of a square with side length 6. If a random point P is chosen randomly inside the square, what is the probability that P lies within all four circles?

Proposed by Bradley Guo.

Answer: $\boxed{0.06082515167823311 89}$

Solution: Let the square be $ABCD$, the midpoint of AB be M, the center of the square be O, and the intersection between the circle centered at A and B be Q. Since $AQ = 5$ and $AM = 3$, we know $QM = 4$ and therefore $OQ = 1$. At this point, we can approximate the area of the region of interest as a square with diagonal length 2, which gives an answer of $\frac{2}{18} \approx 0.0556$. The true answer is approximately $\frac{2.1897}{36} \approx 0.0608$

28 Mr. Rose's evil cousin, Mr. Caulem, has teaches a class of three hundred bees. Every week, he tries to disrupt Mr. Rose's 4th period by sending three of his bee students to fly around and make human students panic. Unfortunately, no pair of bees can fly together twice, as then Mr. Rose will become suspicious and trace them back to Mr. Caulem. What's the largest number of weeks Mr. Caulem can disrupt Mr. Rose's class?

Proposed by Nathan Cho.

Answer: $\boxed{14900}$

Solution: N/A

29 Two blind brothers Beard and Bored are driving their tractors in the middle of a field facing north, and both are 10 meters west from a roast turkey. Beard, can turn exactly $0.7°$ and Bored can turn exactly $0.2°$ degrees. Driving at a consistent 2 meters per second, they drive straight until they notice the smell of the turkey getting farther away, and then turn right and repeat until they get to the turkey.

Suppose Beard gets to the Turkey in about 818.5 seconds. Estimate the amount of time it will take Bored.

Proposed by Nathan Cho.

Answer: $\boxed{2864.786}$

Solution: Both brothers basically spiral in towards the center. If they are currently x meters from the turkey, and they can turn at a θ angle, then they will end up $x \cos \theta$

meters away from the turkey and travel $x \sin\theta$ meters. Therefore, in the end, the brother will travel

$$10\sin\theta + 10\cos\theta\sin\theta + 10\cos^2\theta\sin\theta + ... = \frac{10\sin\theta}{1-\cos\theta} = 10\cot\theta/2,$$

and dividing by the speed will yield the amount of time it will take. If we could just plug in $\theta = 0.2°$, we would be done. In fact, this yields about 2864.786 after dividing by 2 to account for the speed, when plugged into a calculator.

We are given that

$$5\cot 0.35° \approx 818.5.$$

Since $y = x$ approximates $\tan x$ for values of x close to zero, we can say that dividing the angle by 10 when it is close to zero will approximately multiply the entire thing by 10. Therefore, we can reasonably deduce that

$$5\cot 0.1° = \frac{5}{\tan(0.1°)} \approx 3.5\frac{5}{\tan(0.35°)} = 0.35 \cdot 5\cot 0.35° \approx 2864.75.$$

Noting that $\tan x > x$ for the x we are concerned with tells us that our fraction increases by more than a factor of 3.5, but very close to a factor of 3.5. By how much more, however, is a harder question to answer.

30 Let a be the probability that 4 randomly chosen positive integers have no common divisor except for 1. Estimate $300a$. Note that the integers $1, 2, 3, 4$ have no common divisor except for 1.

Remark. This problem is asking you to find

$$300 \lim_{n\to\infty} a_n,$$

if a_n is defined to be the probability that 4 randomly chosen integers from $\{1, 2, ..., n\}$ have greatest common divisor 1.

Proposed by Nathan Cho.

Answer: $\boxed{\frac{27000}{\pi^4} \approx 277.181520876...}$

Solution: For any prime number p, we must have that p does not divide any of the numbers. The probability that any number is divisible by p is $1/p$. Hence, the probability that there exists one number not divisible by p is $1 - p^{-4}$.

From here, there are two ways to finish, one of which requires advanced background knowledge, and another that reduces the problem through approximations which gives you full points.

Approximating for full points

We note that when p is big, then p^{-4} is very small. Namely, note that $p > 10$ implies that $p^{-4} < 0.01\%$. It is up to the competitor how much computation they want to do, for the number of points they receive. Surprisingly, only 2 terms are needed to get within 1 of the answer:

$$\left(1 - 2^{-4}\right)\left(1 - 3^{-4}\right)300 = 277.\overline{7},$$

which is enough for full points.

Closed form, with high powered theory

Earlier, we found that our answer is

$$\prod_p \left(1 - p^{-4}\right)$$

$$\left(\prod_p \frac{1}{1 - p^{-4}}\right)^{-1}$$

We note that this is an Euler product, namely, this becomes

$$\begin{aligned}
&\left(\prod_p \frac{1}{1 - p^{-4}}\right)^{-1} \\
&= \prod_p \left(1 - p^{-4}\right) \\
&= \left(\prod_p \sum_{k=0}^{\infty} \frac{1}{p^{4k}}\right)^{-1} \\
&= \left(\left(\frac{1}{1} + \frac{1}{2^4} + \frac{1}{2^8} + \cdots\right)\left(\frac{1}{1} + \frac{1}{3^4} + \frac{1}{3^8} + \cdots\right)\left(\frac{1}{1} + \frac{1}{5^4} + \cdots\right)\cdots\right)^{-1} \\
&= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)^{-1} \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{k^4}\right)^{-1} \\
&= \frac{1}{\zeta(4)} = \frac{90}{\pi^4}
\end{aligned}$$

Hence, the answer is

$$\frac{27000}{\pi^4} \approx \boxed{277.181520876...}$$