Montgomery Blair Math Tournament
Online Round

June 13-17, 2020

This round consists of **45** questions. You will have from **June 13 to June 17** to complete the round. Point values for questions are based on the number of solves; see the submission site for more details.

Problems **41-45** were released on **June 16** as optional enrichment problems. Please note that these problems will not contribute to the final team score. However, you can still submit answers to these problems and receive feedback for them.

Submissions and Information:

https://online.mbmt.mbhs.edu

1 Chris has a bag with 4 black socks and 6 red socks (so there are 10 socks in total). Timothy reaches into the bag and grabs two socks *without replacement*. Find the probability that he will not grab two red socks.

Proposed by Chris Tong

Solution. $\boxed{\dfrac{2}{3}}$

The probability that the first sock is red is $\frac{3}{5}$ and the probability that the second is red is $\frac{5}{9}$. Therefore, the probability that he does not get a pair of reds is $1 - \frac{3}{5} \cdot \frac{5}{9} = \frac{2}{3}$ $\square$

2 Daniel, Clarence, and Matthew split a \$20.20 dinner bill so that Daniel pays half of what Clarence pays. If Daniel pays \$6.06, what is the ratio of Clarence's pay to Matthew's pay?

Proposed by Henry Ren

Solution. $\boxed{6}$

Since Daniel paid \$6.06, Clarence must have paid double that, or \$12.12. In total, the two paid \$18.18, which means that Matthew must have paid the remaining \$2.02. The answer is thus $\boxed{6}$. $\square$

3 Square $ABCD$ has a side length of 1. Point E lies on the interior of $ABCD$, and is on the line $\overleftrightarrow{AC}$ such that the length of $\overline{AE}$ is 1. Find the shortest distance from point E to a side of square $ABCD$.

Proposed by Chris Tong

Solution. $\boxed{\dfrac{2-\sqrt{2}}{2}}$

Point E is equidistant to sides BC and CD, the sides which it is closest to. Draw the perpendicular, and use the properties of $45-45-90$ triangles to find the desired length to be $\frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$. $\square$

4 Ken has a six sided die. He rolls the die, and if the result is not even, he rolls the die one more time. Find the probability that he ends up with an even number.

Proposed by Gabriel Wu

Solution. $\boxed{\dfrac{3}{4}}$

He has a $\frac{1}{2}$ chance of rolling a even on the first roll, and a $\frac{1}{4}$ chance of rolling a even on the second. Thus, the answer is $\frac{1}{2} + \frac{1}{4}$. $\square$

5 Fuzzy draws a segment of positive length in a plane. How many locations can Fuzzy place another point in the same plane to form a non-degenerate isosceles right triangle with vertices consisting of his new point and the endpoints of the segment?

Proposed by Timothy Qian

Solution. $\boxed{6}$

We perform casework on whether the side is a leg or hypotenuse of the isosceles right triangle. If it is a leg, then there are 4 possibilities for the other point. If it is a hypotenuse, there are 2 possibilities. Thus the total is $2 + 4 = \boxed{6}$. □

6 Given that $\sqrt{10} \approx 3.16227766$, find the largest integer n such that $n^2 \leq 10,000,000$.

Proposed by Jacob Stavrianos

Solution. $\boxed{3162}$

The condition is equivalent to $n \leq 1000 \cdot \sqrt{10}$, which is just shifting the decimal place on $\sqrt{10}$ by 3 places, yielding $n = 3162$. □

7 Let $S = \{1, 2, 3, ..., 12\}$. How many subsets of S, excluding the empty set, have an even sum but not an even product?

Proposed by Gabriel Wu

Solution. $\boxed{31}$

If a subset of S does not have an even product, it means that it does not contain any even values. To have an even sum, the subset must then have an even amount of odd elements.There are 6 possible odd elements: $1, 3, 5, 7, 9, 11$. Thus, the answer is $\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$. □

8 Let $\triangle ABC$ be inscribed in circle O with $\angle ABC = 36°$. D and E are on the circle such that $\overline{AD}$ and $\overline{CE}$ are diameters of circle O. List all possible positive values of $\angle DBE$ in degrees in order from least to greatest.

Proposed by Ambrose Yang

Solution. $\boxed{36°, 144°}$

Let O be the center of the circle. We have that $\angle AOC = 36° \cdot 2 = 72°$. Since D, E are opposite of A, C, respectively, we have that $\angle DOE = \angle AOC = 72°$. Thus $\angle DBE$ is either equal to $36°$ or $180° - 36° = 144°$, depending on whether B is on minor or major arc $\widehat{\text{DE}}$. □

9 Consider a regular pentagon $ABCDE$, and let the intersection of diagonals $\overline{CA}$ and $\overline{EB}$ be F. Find $\angle AFB$.

Proposed by Justin Chen

Solution. $\boxed{108^\circ}$

Because $ABCDE$ is regular, $\overline{EB}$ is parallel to $\overline{DC}$ and $\overline{AC}$ is parallel to $\overline{ED}$. Therefore $EDCF$ is a parallelogram. $\angle EFC = 108^\circ$, so $\angle AFB = \angle EFC = 108^\circ$ by equal angles of a parallelogram. □

______ **10** Mr. Squash bought a large parking lot in Utah, which has an area of 600 square meters. A car needs 6 square meters of parking space while a bus needs 30 square meters of parking space. Mr. Squash charges \$2.50 per car and \$7.50 per bus, but Mr. Squash can only handle at most 60 vehicles at a time. Find the ordered pair (a, b) where a is the number of cars and b is the number of buses that maximizes the amount of money Mr. Squash makes.

Proposed by Nathan Cho

Solution. $\boxed{(50, 10)}$

Setting up a system of inequalities, we find that we want to maximize $2.50 \cdot a + 7.50 \cdot b$ under the constraint that a, b are nonnegative integers, and $6a + 30b \leq 600, a + b \leq 60$. Since we only care about what a and b are, we can think of this as maximizing $a + 3b$. We simplify the constraints to $a + b \leq 60$, $a + 5b \leq 100$. Let $k = a + 3b$. The equations can be rewritten as $k - 2b \leq 60 \Rightarrow k \leq b + 60$, and $k \leq 100 - 3b$. So we have $k \leq \min(b + 60, 100 - 3b)$. Since the first argument is increasing while the second argument is decreasing, the maximum possible value of the right hand side is achieved when the two arguments are equal. This happens at $b = 10$. This maximum value of k is achievable by setting $a = 50$, which gives us our desired pair. □

______ **11** There are 8 distinct points on a plane, where no three are collinear. An ant starts at one of the points, then walks in a straight line to each one of the other points, visiting each point exactly once and stopping at the final point. This creates a trail of 7 line segments. What is the maximum number of times the ant can cross its own path as it walks?

Proposed by Gabriel Wu

Solution. $\boxed{15}$

The first two segments cannot cause any intersections. After that, each segment can intersect every previous segment, besides the one that comes just before it. The i^{th} segment can thus create $i-2$ intersections. The answer is $0+0+1+2+3+4+5 = 15$. □

______ **12** Find the number of ways to partition $S = \{1, 2, 3, \ldots, 2020\}$ into two disjoint sets A and B with $A \cup B = S$ so that if you choose an element a from A and an element b from B, $a + b$ is never a multiple of 20. A or B can be the empty set, and the order of A and B doesn't matter. In other words, the pair of sets (A, B) is indistinguishable from the pair of sets (B, A).

Proposed by Timothy Qian

Solution. $\boxed{1024}$

Consider the residues mod 20. There are 20 of them. $x, 20 - x \pmod{20}$ must be in the same group. Thus we get $9 + 2 = 11$ groups of residues to split into two sets. Thus the answer is 2^{10}, as we don't care about the order of the sets. □

13 How many ordered pairs of positive integers (a, b) are there such that a right triangle with legs of length a, b has an area of p, where p is a prime number less than 100?

Proposed by Joshua Hsieh

Solution. $\boxed{99}$

The right triangle must have a side with length 1 or 2. So for each choice of p, there are 2 such triangles. Since there are 25 primes less than 100, we initially have an answer of $2 \cdot 50$, where the 2 is to account for the fact that (a, b) are ordered. However, we subtract 1 since we overcount $(2, 2)$ twice, so our answer is 99. □

14 Mr. Schwartz has been hired to paint a row of 7 houses. Each house must be painted red, blue, or green. However, to make it aesthetically pleasing, he doesn't want any three consecutive houses to be the same color. Find the number of ways he can fulfill his task.

Proposed by Daniel Monroe

Solution. $\boxed{1344}$

Let $f(n)$ be the number of ways Mr. Schwartz can paint a row of n houses with the given restrictions. Then $f(1) = 3$ and $f(2) = 9$. For all $n \geq 3$, the last two houses can either be the same color or different colors. If they are different colors, there are $2f(n-1)$ ways to color the houses because for each coloring of $n - 1$ houses, the next house can be either of the two colors that the second to last house isn't. If the last two houses are the same color, then they must be a different color than the third to last house (otherwise the last three houses would all be the same color). This contributes $2f(n-2)$ ways. Thus, $f(n) = 2f(n-1) + 2f(n-2)$. Applying this recursive function gets us $f(3) = 24, f(4) = 66, f(5) = 180, f(6) = 492, f(7) = \boxed{1344}$. □

15 Bread draws a circle. He then selects four random distinct points on the circumference of the circle to form a convex quadrilateral. Kwu comes by and randomly chooses another 3 distinct points (none of which are the same as Bread's four points) on the circle to form a triangle. Find the probability that Kwu's triangle does not intersect Bread's quadrilateral, where two polygons intersect if they have at least one pair of sides intersecting.

Proposed by Nathan Cho

Solution. $\boxed{\frac{1}{5}}$

We note that locations of the points themselves don't affect the intersection between the quadrilateral and the triangle, only their relative ordering around the circle does. Note that the triangle and quadrilateral do not intersect if and only if the three points of the triangle are placed consecutively around the circle. In other words, all three vertices of the triangle fall between two adjacent vertices of the quadrilateral. If we fix one of the points of the quadrilateral, we can reduce the problem to finding the probability that a random permutations of the string "QQQTTT" has three T's in a row. Out of all $\binom{6}{3}$ permutations, only 4 work. Thus, the probability that the triangle and quadrilateral do not intersect is $\frac{4}{\binom{6}{3}} = \boxed{\frac{1}{5}}$. □

16 What is the largest integer n with no repeated digits that is relatively prime to 6? Note that two numbers are considered relatively prime if they share no common factors besides 1.

Proposed by Jacob Stavrianos

Solution. $\boxed{987654301}$

The condition is equivalent to neither $2|n$ nor $3|n$.

We initially note that n can contain at most every digit exactly once. However, applying the rule for divisibility by 3, we find that $0+1+2+3+4+5+6+7+8+9 = 45$ is divisible by 3, so n must be missing a digit. Also, n cannot end in a 0 or 2, otherwise 2 would divide n.

To maximize n, we claim that there exists a 9-digit n starting with 9876543. The remaining numbers are $2, 1, 0$, of which one must not be included. We see that n must end in 1 for it to be odd, and if 0 was excluded then $3|n$. Thus, 2 is excluded, and $n = \boxed{987654301}$. □

17 $\triangle KWU$ is an equilateral triangle with side length 12. Point P lies on minor arc $\widehat{\text{WU}}$ of the circumcircle of $\triangle KWU$. If $\overline{KP} = 13$, find the length of the altitude from P onto $\overline{WU}$.

Proposed by Bradley Guo

Solution. $\boxed{\frac{25\sqrt{3}}{24}}$

Let X be the point opposite of point K on the circle (KWU). We note that the circumradius of an equilateral triangle with side length 12 is $\frac{12}{\sqrt{3}} = 4\sqrt{3}$, so $\overline{KX} = 8\sqrt{3}$ since it is a diameter. Let the foot of the altitude from P to $\overline{KX}$ be Y, and let the intersection of $\overline{KX}$ and $\overline{WU}$ be Z. Note that Z is the midpoint of $\overline{WU}$ by symmetry. Note that $\overline{KP} \perp \overline{WU}$. So we simply want to find $\overline{YZ}$. We have that $\overline{KZ} = \frac{12\sqrt{3}}{2} = 6\sqrt{3}$ because it is the height of an equilateral triangle. We have that $\triangle KPX$ is right, so

$PX = \sqrt{23}$ by the Pythagorean Theorem. Thus $PY = \frac{\sqrt{23}\cdot 13}{8\sqrt{3}}$ since it is the height of $\triangle PKX$. By Pythagorean theorem, $\overline{XY} = \frac{23\sqrt{3}}{24}$. Thus we have our final answer is $\overline{YZ} = 8\sqrt{3} - \overline{KZ} - \overline{XY} = \frac{25\sqrt{3}}{24}$.

$\square$

______ **18** Let w, x, y, z be integers from 0 to 3 inclusive. Find the number of ordered quadruples of (w, x, y, z) such that $5x^2 + 5y^2 + 5z^2 - 6wx - 6wy - 6wz$ is divisible by 4.

Proposed by Timothy Qian

Solution. $\boxed{32}$

You can complete the square after taking $\pmod 4$ to get that $(x-w)^2 + (y-w)^2 + (z-w)^2 \equiv 3w^2 \pmod 4$. So we can do casework on the parities of w, x, y, z. If we consider the case of w even, we clearly must have x, y, z all be even, and if we consider the case of w odd, we clearly must have x, y, z all be even. Thus there are $4 \cdot 2^3 = 32$ quadruples. $\square$

______ **19** In a regular hexagon $ABCDEF$ of side length 8 and center K, points W and U are chosen on $\overline{AB}$ and $\overline{CD}$ respectively such that $\overline{KW} = 7$ and $\angle WKU = 120°$. Find the area of pentagon $WBCUK$.

Proposed by Bradley Guo

Solution. $\boxed{32\sqrt{3}}$

Note that since the hexagon satisfies 120° rotational symmetry about K, pentagon $WBCUK$ rotated three times about K, each time by 120°, forms the complete hexagon $ABCDEF$. Thus, the area is just $\frac{1}{3}$ the area of hexagon $ABCDEF$, which we can calculate. Therefore, the answer is $32\sqrt{3}$. $\square$

______ **20** Sam colors each tile in a 4 by 4 grid white or black. A coloring is called *rotationally symmetric* if the grid can be rotated 90, 180, or 270 degrees to achieve the same pattern. Two colorings are called *rotationally distinct* if neither can be rotated to match the other. How many rotationally distinct ways are there for Sam to color the grid such that the colorings are *not* rotationally symmetric?

Proposed by Gabriel Wu

Solution. $\boxed{16320}$

The total number of ways for Sam to color the grid is 2^{16}. Among these ways, 2^8 are rotationally symmetric. Among the remaining ways, each distinct way is counted 4 times. Thus the answer is $\frac{2^{16}-2^8}{4}$. $\square$

21 Matthew Casertano and Fox Chyatte make a series of bets. In each bet, Matthew sets the stake (the amount he wins or loses) at half his current amount of money. He has an equal chance of winning and losing each bet. If he starts with \$256, find the probability that after 8 bets, he will have at least \$50.

Proposed by Jeffrey Tong

Solution. $\boxed{\frac{163}{256}}$

"Each time Matthew wins a bet, his balance will be multiplied by $\frac{3}{2}$, and each time he loses, it will be multiplied by $\frac{1}{2}$. Therefore, if he wins n bets, his final balance will be $256 \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{8-n}$, which simplifies to 3^n. Since we want $3^n \geq 50$, we need $n \geq 4$.

To find the probability, we now divide the total number of ways to achieve 4 or more wins by the total number of possible outcomes, 2^8. The number of ways to get exactly k wins is $\binom{8}{k}$, so this gives $\frac{\binom{8}{4}+\binom{8}{5}+\binom{8}{6}+\binom{8}{7}+\binom{8}{8}}{2^8} = \frac{70+56+28+8+1}{256} = \frac{163}{256}$. □

22 Find the product of all positive real solutions to the equation $x^{-x} + x^{\frac{1}{x}} = \frac{2021}{2020}$

Proposed by Gabriel Wu

Solution. $\boxed{1}$

Turn the LHS into $x^{\frac{1}{x}} + (\frac{1}{x})^x$. Then, you know that if x is a solution, then 1/x must also be a solution because plugging in $\frac{1}{x}$ gets you the same expression. Thus, the product of all positive solutions that work is 1. □

23 Let $ABCD$ be a cyclic quadrilateral so that $\overline{AC} \perp \overline{BD}$. Let E be the intersection of $\overline{AC}$ and $\overline{BD}$, and let F be the foot of the altitude from E to $\overline{AB}$. Let $\overline{EF}$ intersect $\overline{CD}$ at G, and let the foot of the perpendiculars from G to $\overline{AC}$ and $\overline{BD}$ be H, I respectively. If $\overline{AB} = \sqrt{5}, \overline{BC} = \sqrt{10}, \overline{CD} = 3\sqrt{5}, \overline{DA} = 2\sqrt{10}$, find the length of $\overline{HI}$.

Proposed by Timothy Qian

Solution. $\boxed{\frac{3\sqrt{5}}{2}}$

We evidently have that $\triangle AEB$ and $\triangle CED$ are both right triangles and similar by cyclic quadrilaterals. Let $a = \angle BAE$. We thus have $\angle BEF = a$. However, we also have $\angle EDC = a$ by the similar triangles, so we have that $\angle BEF = \angle DEG = \angle CDE = a$. This implies that $\triangle DGE$ is isosceles with $\overline{DG} = \overline{GE}$. We can similarly find $\overline{GE} = \overline{GC}$. This shows that G is the midpoint of $\overline{DC}$. Note that $EHGI$ is a rectangle since we have three of its angles are right angles. Thus we have $\overline{HI} = \overline{EG} = \overline{DG} = \frac{\overline{DC}}{2} = \frac{3\sqrt{5}}{2}$. □

24 Nashan randomly chooses 6 positive integers a, b, c, d, e, f. Find the probability that $2^a + 2^b + 2^c + 2^d + 2^e + 2^f$ is divisible by 5.

Proposed by Bradley Guo

Solution. $\boxed{\dfrac{205}{1024}}$

For any integer n, if $n \equiv 0 \mod 4$, then $2^n \equiv 1 \mod 5$, if $n \equiv 1 \mod 4$, then $2^n \equiv 2 \mod 5$, if $n \equiv 2 \mod 4$, then $2^n \equiv 4 \mod 5$, if $n \equiv 3 \mod 4$, then $2^n \equiv 3 \mod 5$. Thus, $2^a, 2^b, 2^c, 2^d$ can each be $1, 2, 3$ or $4 \mod 5$ with equal probability. We can establish a recurrence: $p(n) = (1 - p(n-1)) \cdot \frac{1}{4}$ where $p(n)$ is the probability that the sum of 2 taken to the power of n random integers is divisible by 5. We are looking for $p(6)$ and $p(1) = 0$, so the answer is $\boxed{\dfrac{205}{1024}}$. □

25 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the sum of all positive integer solutions to

$$\left\lfloor \frac{n^3}{27} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor^3 = 10.$$

Proposed by Jason Hsu

Solution. $\boxed{18}$

Let $x = 3a + b$ for nonnegative integers a, b such that $0 \le b \le 2$. We have that this expression can be rewritten as $\left\lfloor a^3 + a^2 b + \frac{9ab^2+1}{27} \right\rfloor - a^3 = 10$. This means that $\left\lfloor a^2 b + \frac{9ab^2+1}{27} \right\rfloor = 10$. We can now perform casework on b.

Case 1: $b = 0$. We have that $\left\lfloor \frac{1}{27} \right\rfloor = 10$, which is false.

Case 2: $b = 1$. We have $\left\lfloor a^2 + \frac{9a+1}{27} \right\rfloor = 10$. a has to be less than or equal to 3, and the only value that works is $a = 3$. Thus we get a solution of $n = 10$.

Case 3: $b = 2$. We have $\left\lfloor 2a^2 + \frac{36a+1}{27} \right\rfloor = 10$. We have $a \le 2$, and the only solution that works is $a = 2$. Thus we another solution of $n = 8$.

These are our only solutions, so our solution set is $8, 10 \Rightarrow 18$. □

26 Let $\triangle MBT$ be a triangle with $\overline{MB} = 4$ and $\overline{MT} = 7$. Furthermore, let circle ω be a circle with center O which is tangent to $\overline{MB}$ at B and $\overline{MT}$ at some point on segment $\overline{MT}$. Given $\overline{OM} = 6$ and ω intersects $\overline{BT}$ at $I \neq B$, find the length of $\overline{TI}$.

Proposed by Chad Yu

Solution. $\boxed{\dfrac{27\sqrt{641}}{641}}$

Let ω be tangent to MT at B'. We know that $MB = MB' = 4$, so B'T = 3. Also, we know that OM bisects $\angle BMT$, so $\cos \angle BMT = 2\cos^2(\angle BMO) - 1 = 2 \cdot \left(\frac{2}{3}\right)^2 - 1 = -\frac{1}{9}$. Thus we have that $BT^2 = MB^2 + MT^2 - 2 \cdot MB \cdot MT \cdot \cos \angle BMT \Rightarrow BT = \frac{\sqrt{641}}{3}$. Lastly, by Power of a Point, we have that $TI = \frac{B'T^2}{BT} = \frac{27}{\sqrt{641}}$. □

27 The perfect square game is played as follows: player 1 says a positive integer, then player 2 says a strictly smaller positive integer, and so on. The game ends when someone says 1; that player wins if and only if the sum of all numbers said is a perfect square. What is the sum of all n such that, if player 1 starts by saying n, player 1 has a winning strategy? A winning strategy for player 1 is a rule player 1 can follow to win, regardless of what player 2 does. If player 1 wins, player 2 must lose, and vice versa. Both players play optimally.

Proposed by Jacob Stavrianos

Solution. $\boxed{9}$

We note $n = 1$ and $n = 2$ are wins for player 1, and consider the case where $n > 2$:

For player 2, consider the strategy of "say 2 on my first move". Player 1 is then forced to say 1 and wins iff $n + 3$ is a perfect square. Thus, $n + 3$ must be a perfect square for player 1 to have a winning strategy.

Now, consider the following strategy for player 2: "say the smallest available number that brings the sum to 3 minus a perfect square. Then, player 1 can't say 2, and I'll say 2 on my next turn." This strategy wins for player 2 whenever any such number is available. Setting $n + 3 = a^2$, we check when such a number exists:

$$((a + 1)^2 - 3) - (a^2 - 3) < a^2 - 3$$

$$2a + 1 < a^2 - 3$$
$$a^2 > 2(a + 2)$$

We manually verify that this is true for all $a > 3$, so player 1's strategy is winning for $a \leq 3$. The answer is thus $1 + 2 + 6 = \boxed{9}$. □

28 Consider the system of equations

$$a + 2b + 3c + \ldots + 26z = 2020$$

$$b + 2c + 3d + \ldots + 26a = 2019$$

$$\vdots$$

$$y + 2z + 3a + \ldots + 26x = 1996$$
$$z + 2a + 3b + \ldots + 26y = 1995$$

where each equation is a rearrangement of the first equation with the variables cycling and the coefficients staying in place. Find the value of

$$z + 2y + 3x + \cdots + 26a.$$

Proposed by Joshua Hsieh

Solution. $\boxed{1995}$

Adding the first expression and our desired expression, we obtain

$$27(a+b+c+\ldots+z)$$

Summing the given equations, we have

$$\frac{26(27)}{2}(a+b+c+\ldots+z)=\frac{26(2020+1995)}{2}$$

which means that $a+b+c+\ldots+z=\frac{4015}{27}$.

Thus the desired expression is equal to $27\cdot\left(\frac{4015}{27}\right)-2020=\boxed{1995}$ □

29 The center of circle ω_1 of radius 6 lies on circle ω_2 of radius 6. The circles intersect at points K and W. Let point U lie on the major arc $\widehat{\mathrm{KW}}$ of ω_2, and point I be the center of the largest circle that can be inscribed in $\triangle KWU$. If $KI+WI=11$, find $KI\cdot WI$.

Proposed by Bradley Guo

Solution. $\boxed{13}$

We can easily find that $KW=6\sqrt{3}$ and $\angle KUW=60^\circ$ using equilateral triangles. Since I lies on the angle bisector of $\angle KUW$, $\angle KIW=120^\circ$. Using Law of Cosines on $\triangle KIW$,

$$KI^2+WI^2+KI\cdot WI=108$$

We are given that $KI+WI=11$, so

$$KI^2+WI^2+2\cdot WI\cdot KI=121$$

Thus,

$$KI\cdot WI=\boxed{13}$$

□

30 Let the number of ways for a rook to return to its original square on a 4×4 chessboard in 8 moves if it starts on a corner be k. Find the number of positive integers that are divisors of k. A "move" counts as shifting the rook by a positive number of squares on the board along a row or column. Note that the rook may return back to its original square during an intermediate step within its 8-move path.

Proposed by Bradley Guo

Solution. $\boxed{36}$

Let w_u be the number of ways for a rook to return to the corner after u moves. After $u-2$ moves, the rook will either be at it's original square, along the row and column of the original square, or anywhere else. The number of ways for the first case to occur is w_{u-2}, the number of ways for the second to occur is w_{u-1}, and the number of ways for the third to occur is $6^{n-2} - w_{u-1} - w_{u-2}$. The rook has 6 ways to return after 2 more moves for the first case, 2 way to return for the second, and 2 ways to return for the third. Thus, we find that

$$w_u = 2 \cdot 6^{n-2} + 4 \cdot w_{u-2}$$

We initially have $w_1 = 0, w_2 = 6$. If we calculate w_8, we find that

$$w_8 = 2 \cdot 6^6 + 8 \cdot 6^4 + 32 \cdot 6^2 + 64 \cdot 6 = 6 \cdot 64 \cdot (3^5 + 3^3 + 3 + 1) = 6 \cdot 64 \cdot 274 = 2^8 \cdot 3 \cdot 137$$

This thus has $\boxed{36}$ factors. □

31 Consider the infinite sequence $\{a_i\}$ that extends the pattern

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

Formally, $a_i = i - T(i)$ for all $i \geq 1$, where $T(i)$ represents the largest triangular number less than i (triangle numbers are integers of the form $\frac{k(k+1)}{2}$ for some nonnegative integer k). Find the number of indices i such that $a_i = a_{i+2020}$.

Proposed by Gabriel Wu

Solution. $\boxed{2702}$

Let i be called a *peak* if $a_{i+1} = 1$. Notice that whenever $a(i) = a(j)$, $j - i$ can be written as the sum of the values of the peaks between i and j, which must be consecutive positive integers. This can be found through intuition and easily shown. In this case, we want to write $2020 = x + (x+1) + (x+2) + \cdots + (x+n)$. Given such a representation, a_i can be any number from 1 to x, and n has to be a nonnegative integer. So we have reduced the problem to finding the sum of all x over all pairs of (x, n) that have $2020 = x + (x+1) + (x+2) + \cdots + (x+n)$.

If n is even, then the average of $x, x+1, \ldots, x+n$ must be an integer, so the number of terms must be a factor of 2020 that allows x to be a positive integer. Checking these cases gets us $(x, n) = (402, 4)$ or $(2020, 0)$. If n is odd, then the average of the terms is a fraction with a denominator of 2. Thus, the number of terms is a multiple of 8. Checking some more cases, the only solutions that work are $(x, n) = (249, 7)$ or $(31, 39)$. The sum of all valid x is $402 + 2020 + 249 + 31 = 2702$. □

____________ **32** Let the *square decomposition* of a number be defined as the sequence of numbers given by the following algorithm. Given a positive integer n, add the largest possible perfect square that is less than or equal to n to a sequence, and then subtract that number from n. Repeat as many times as necessary until your current n is 0. So for example, the square decomposition of 60 would be $49, 9, 1, 1$. Define the size of a square decomposition to be the number of numbers in the sequence. Say that the maximal size of a square decomposition of a number in the range $[1, 2020]$ is m. Find the largest number in the range $[1, 2020]$ that has a square decomposition of size m.

Proposed by Timothy Qian

Solution. $\boxed{2015}$

Let $f(n)$ be the function that outputs the length of the square decomposition of a n, and let $s(n)$ output the largest square less than or equal to n. Then it's easy to see that $f(n) = f(n - s(n)) + 1$. Note that the smallest square greater than 2020 is 2025. Thus m can be at most $1 + f(x)$ for some x in the range $[0, 2024 - 44^2]$. Thus, this gives us an easy way to find what m is, and after applying this several times, we can deduce m to be 6. Now we want the largest number with a square decomposition of size 6. Evidently we want it to be in the range of $[44^2, 2020]$. This reduces to finding the largest number in the range of $[0, 84]$ with a square decomposition of size 5. We can repeat this reduction process to eventually extract our answer of 2015. □

____________ **33** Circle ω_1 with center K of radius 4 and circle ω_2 of radius 6 intersect at points W and U. If the incenter of $\triangle KWU$ lies on circle ω_2, find the length of $\overline{WU}$. (Note: The incenter of a triangle is the intersection of the angle bisectors of the angles of the triangle)

Proposed by Bradley Guo

Solution. $\boxed{\dfrac{24\sqrt{13}}{13}}$

Let I be the incenter of $\triangle KWU$. Let $x = \angle WKU$. By angle chasing (or properties of the incenter), we know that $\angle WIU = 90^\circ + \frac{x}{2}$. Let O be the center of ω_2. By inscribed angles, we have that $\angle WOU = 180^\circ - x$. However, this implies that $KWOU$ is cyclic. Since this quadrilateral is symmetric about $\overline{KO}$, we must have $\overline{KO}$ is a diameter of the circumcircle of $KWOU$, so we have $\angle KWO = 90^\circ$. Thus we have that WU is two times the length of the altitude from W to $\overline{KO}$. This is easy to calculate since $\triangle KWO$ is a right triangle with $\overline{KW} = 4, \overline{WO} = 6$, and we get $\boxed{\dfrac{24\sqrt{13}}{13}}$ as our answer. □

____________ **34** Let a set S of n points be called *cool* if:

- All points lie in a plane
- No three points are collinear

- There exists a triangle with three distinct vertices in S such that the triangle contains another point in S strictly inside it

Define $g(S)$ for a cool set S to be the sum of the number of points strictly inside each triangle with three distinct vertices in S. Let $f(n)$ be the minimal possible value of $g(S)$ across all cool sets of size n. Find

$$f(4) + \cdots + f(2020) \pmod{1000}$$

Proposed by Timothy Qian

Solution. $\boxed{153}$

We find the explicit formula for $f(n)$. First we will solve this for the case of a convex $n-1$ gon and 1 point inside. Call a triangulation from a vertex the triangulation formed by drawing all diagonals from that vertex. Across all vertex triangulations of the $n-1$ gon, there is a triangle in that triangulation that contains that point. However, each pair of vertex triangulations can only overlap at most one triangle, namely a triangle that has two sides on the $n-1$ gon. At most two of these triangles can contain the one point inside, thus we have our minimal bound of $n-3$. It is easily verifiable that this is achievable.

Now we consider the general case. Consider the convex hull of the n points, say there are a on it, and b on the inside. A very rough lower bound is $(a-2) \cdot b$ under the constraints $a \geq 3, b \geq 1, a+b=n$ by using the simpler case before. The minimum is evidently achieved at $n-3$, where $a = n-1, b = 1$. This achieved by the construction in the smipler case. Thus our $f(n) = n-3$, and our answer is $1+2+\cdots+2017 = 2035153 \Rightarrow \boxed{153}$. □

35 Tim has a multiset of positive integers. Let c_i be the number of occurrences of numbers that are *at least* i in the multiset. Let m be the maximum element of the multiset. Tim calls a multiset *spicy* if $c_1, \ldots, c_m$ is a sequence of strictly decreasing powers of 3. Tim calls the *hotness* of a spicy multiset the sum of its elements. Find the sum of the hotness of all spicy multisets that satisfy $c_1 = 3^{2020}$. Give your answer $\pmod{1000}$. (Note: a multiset is an unordered set of numbers that can have repeats)

Proposed by Timothy Qian

Solution. $\boxed{576}$

Let the sorted elements of the multiset be $a_1, \ldots, a_n$. Draw a dot plot, where in the ith column from the left, we draw a_i vertical dots. Then the hotness of a spicy set is the number of dots drawn. However, the condition implies that each horizontal row has a number of dots equivalent to a power of 3. Moreover, each row has a distinct power of 3 number of dots, and this value decreases from 3^{2020} going upwards. Thus a characterization of the possible hotness of any spicy set are the numbers with only 1's and 0's in their base 3 representation if they have exactly 2021 digits. Let $n = 2020$ for simplicity. We can compute this for each bit in the ternary representation across all hotness values to be

$$3^n \cdot 2^n + (1 + 3 + \cdots + 3^{n-1}) \cdot 2^{n-1} = 45 \cdot 6^{2018} - 2^{2018}$$

The answer can thus be computed to be 576. □

36 $ABCD$ is a rectangle $\overline{AB} = 5\sqrt{3}$, $\overline{AD} = 30$. Extend $\overline{BC}$ past C and construct point P on this extension such that $\angle APD = 60°$. Point H is on $\overline{AP}$ such that $\overline{DH} \perp \overline{AP}$. Find the length of $\overline{DH}$.

Proposed by Kevin Wu

Solution. $\boxed{\dfrac{15\sqrt{6} - 15\sqrt{2}}{2}}$

Reflect D over C to point X. We have $\overline{XD} = 10\sqrt{3}$, $\overline{DA} = 30$. Note that $30 = 10\sqrt{3} \cdot \sqrt{3}$, and since $\angle D = 90°$, we have that $\triangle DAX$ is a $30° - 60° - 90°$ triangle, with $\angle DXA = 60°$. By the inscribed angles theorem, this implies that $DAXP$ is cyclic since we have $\angle DXA = \angle APD$. Note that we also have that $PD = PX$ by symmetry, thus AP is an angle bisector of $\angle DAX$. We have that $\angle DAX = 30°$, so $\angle PAD = 15°$.

Thus we have that $DH = DA \cdot \sin 15° = \boxed{\dfrac{15\sqrt{6} - 15\sqrt{2}}{2}}$ □

37 Fuzzy likes isosceles trapezoids. He can choose lengths from $1, 2, \ldots, 8$, where he may choose any amount of each length. He takes a multiset of three integers from $1, \ldots, 8$. From this multiset, one length will become a base length, one will become a diagonal length, and one will become a leg length. He uses each element as either a diagonal, leg, or base length exactly once. Fuzzy is happy if he can use these lengths to make an isosceles trapezoid such that the undecided base has nonzero rational length. How many multiset choices can he make? (Multisets are unordered)

Proposed by Timothy Qian

Solution. $\boxed{62}$

Let Fuzzy choose the multiset a, b, c, where these are the leg, diagonal, and base length respectively. Note that a, b, c must be the side lengths of a triangle, as the diagonal, leg length, and base forms a triangle. The isoceles trapezoid is cyclic, so we can apply Ptolemy's theorem to get $b^2 = c^2 + ad$, where d is the final base length. d will always be rational, so we can ignore the rational condition now. Given that a, b, c are the sides of a triangle, the only way we could have a, b, c not be able to form an isosceles trapezoid with rational bases in the matter described is if $a = b = c$. Thus, we simply want to count the number of noncongruent triangles with integer side lengths of at most 8, and subtract out 8 for the 8 equilateral triangles. This yields an answer of $70 - 8 = 62$. □

38 Consider $\triangle ABC$ with circumcenter O and $\angle ABC$ obtuse. Construct A' as the reflection of A over O, and let P be the intersection of $\overline{A'B}$ and $\overline{AC}$. Let P' be the intersection of the circumcircle of (OPA) with $\overline{AB}$. Given that the circumdiameter of $\triangle ABC$ is 25, $\overline{AB} = 7$, and $\overline{BC} = 15$, find the length of PP'.

Proposed by Kevin Wu

Solution. $\boxed{\frac{5\sqrt{37}}{4}}$

Note that $\overline{AA'} = 25$ since it is a circumdiameter. By properties of diameters, $\triangle AA'B$ is right, so we have by the Pythagorean Theorem that $\overline{BA'} = 24$. Let $x = A'C$. We apply Ptolemy's theorem on $ABCA'$ to get that $7x + 15 \cdot 25 = \sqrt{625 - x^2} \cdot 24$. You can solve the resulting quadratic equation (it simplifies nicely) to get $x = 15$ (we discard the negative solution). However, now note that $\overline{BC} = \overline{CA'}$. This implies that $\overline{AC}$ is the angle bisector of $\angle BAA'$. Thus we have that since $AP'PO$ is cyclic, $\overline{PP'} = \overline{PO}$. Note that $\left(\frac{25}{2}\right)^2 - \overline{PO}^2 = Pow(P)$, where Pow is the function for power of a point. We can compute $\overline{AC} = 20$ by the Pythagorean Theorem on right triangle $\triangle ACA'$. Now, we want to find $\overline{AP}$. Let $x = BP$. Then by the Angle Bisector Theorem applied on $\triangle A'AB$, we have that $\frac{7}{x} = \frac{25}{24-x}$. This yields $x = \frac{21}{4}$. Applying the Pythagorean Theorem on $\triangle ABP$ yields $\overline{AP} = \frac{45}{4}$, and subtraction yields $\overline{PC} = \frac{35}{4}$. Thus, we can compute the power of P, and we get that $\overline{PO}^2 = \frac{625}{4} - \frac{35}{4} \cdot \frac{45}{4}$, which yields an answer of $\overline{PP'} = \overline{PO} = \boxed{\frac{5\sqrt{37}}{4}}$. □

39 Let $f(x) = \sqrt{4x^2 - 4x^4}$. Let A be the number of real numbers x that satisfy

$$f(f(f(\ldots f(x)\ldots))) = x,$$

where the function f is applied to x 2020 times. Compute $A \pmod{1000}$.

Proposed by Timothy Qian

Solution. $\boxed{576}$

Note that $f(x) = |2x|\sqrt{1-x^2}$. The domain of this function is $-1 \le x \le 1$; thus we can let $x = \sin(\theta)$ for some θ. We restrict $\theta \in [0, \frac{\pi}{2}]$ since we only care about the value of $\sin(\theta)$. Then, $f(x) = |2\cos(\theta)\sin(\theta)| = |\sin(2\theta)|$. Therefore, $f^n(x) = |\sin(2^n\theta)|$, so we want to find the number of solutions to

$$|\sin\left(2^{2020}\theta\right)| = \sin(\theta)$$

Thus all that remains is to find the number of solutions to θ in the range $[0, \frac{\pi}{2}]$. $|\sin(2^{2020}\theta)|$ is simply a sine function reflected over the y-axis whenever the value becomes negative. Thus, the period of $|\sin(2^{2020}\theta)|$ is $\frac{\pi}{2^{2020}}$, so there are 2^{2019} periods in the range $[0, \frac{\pi}{2}]$. For each period of the function, $\sin(x)$ will intersect $|\sin(2^{2020}\theta)|$

exactly twice if $x \in [0, \frac{\pi}{2}]$, so the final answer is $2 \cdot 2^{2019} = 2^{2020}$. To compute this (mod 1000), we can use the Chinese Remainder Theorem to get a final answer of $\boxed{576}$. ☐

40 Wu starts out with exactly one coin. Wu flips every coin he has *at once* after each year. For each heads he flips, Wu receives a coin, and for every tails he flips, Wu loses a coin. He will keep repeating this process each year until he has 0 coins, at which point he will stop. The probability that Wu will stop after exactly five years can be expressed as $\frac{a}{2^b}$, where a, b are positive integers such that a is odd. Find $a + b$.

Proposed by Bradley Guo

Solution. $\boxed{71622400}$

Let a_n be the probability that Wu will stop after exactly n more years if he has 1 coin at the beginning, and let b_n be the probability that he will stop after exactly n years if he has 2 coins at the beginning. We have $a_1 = \frac{1}{2}, b_1 = \frac{1}{4}$. Note that $a_k = \frac{1}{2}b_{k-1}$ for $k > 1$. Now we find a way to calculate b_k in terms of a_i. The probability that Wu will stop after at most k years if he starts with 2 coins is $(a_1 + \cdots + a_k)^2$. To see this, we treat each coin independently, and each coin has a $a_1 + \cdots + a_k$ probability of becoming 0 coins after at most k years. This yields the desired expression. Thus b_k, or the probability that Wu will stop after exactly k years if he starts with 2 coins is

$$b_k(a_1 + \cdots + a_k)^2 - (a_1 + \cdots + a_{k-1})^2 = a_k^2 + 2a_k(a_1 + \cdots + a_{k-1})$$

$$a_{k+1} = a_k \left(\frac{a_k}{2} + a_{k-1} + \ldots a_1 \right)$$

We can use this recurrence to get what we want, which is a_5. This comes out to

$$1521 \cdot (1521 + 9 \cdot 2^9 + 2^{13} + 2^{15}) + 31 = \boxed{71622400}$$

☐