Montgomery Blair Math Tournament
Online Round

June 13-17, 2020

This round consists of **45** questions. You will have from **June 13 to June 17** to complete the round. Point values for questions are based on the number of solves; see the submission site for more details.

Problems **41-45** were released on **June 16** as optional enrichment problems. Please note that these problems will not contribute to the final team score. However, you can still submit answers to these problems and receive feedback for them.

Submissions and Information:

https://online.mbmt.mbhs.edu

______________ **1** Chris has a bag with 4 black socks and 6 red socks (so there are 10 socks in total). Timothy reaches into the bag and grabs two socks *without replacement*. Find the probability that he will not grab two red socks.

______________ **2** Daniel, Clarence, and Matthew split a \$20.20 dinner bill so that Daniel pays half of what Clarence pays. If Daniel pays \$6.06, what is the ratio of Clarence's pay to Matthew's pay?

______________ **3** Square $ABCD$ has a side length of 1. Point E lies on the interior of $ABCD$, and is on the line $\overleftrightarrow{AC}$ such that the length of $\overline{AE}$ is 1. Find the shortest distance from point E to a side of square $ABCD$.

______________ **4** Ken has a six sided die. He rolls the die, and if the result is not even, he rolls the die one more time. Find the probability that he ends up with an even number.

______________ **5** Fuzzy draws a segment of positive length in a plane. How many locations can Fuzzy place another point in the same plane to form a non-degenerate isosceles right triangle with vertices consisting of his new point and the endpoints of the segment?

______________ **6** Given that $\sqrt{10} \approx 3.16227766$, find the largest integer n such that $n^2 \leq 10,000,000$.

______________ **7** Let $S = \{1, 2, 3, ..., 12\}$. How many subsets of S, excluding the empty set, have an even sum but not an even product?

____________ **8** Let $\triangle ABC$ be inscribed in circle O with $\angle ABC = 36°$. D and E are on the circle such that $\overline{AD}$ and $\overline{CE}$ are diameters of circle O. List all possible positive values of $\angle DBE$ in degrees in order from least to greatest.

____________ **9** Consider a regular pentagon $ABCDE$, and let the intersection of diagonals $\overline{CA}$ and $\overline{EB}$ be F. Find $\angle AFB$.

____________ **10** Mr. Squash bought a large parking lot in Utah, which has an area of 600 square meters. A car needs 6 square meters of parking space while a bus needs 30 square meters of parking space. Mr. Squash charges \$2.50 per car and \$7.50 per bus, but Mr. Squash can only handle at most 60 vehicles at a time. Find the ordered pair (a, b) where a is the number of cars and b is the number of buses that maximizes the amount of money Mr. Squash makes.

____________ **11** There are 8 distinct points on a plane, where no three are collinear. An ant starts at one of the points, then walks in a straight line to each one of the other points, visiting each point exactly once and stopping at the final point. This creates a trail of 7 line segments. What is the maximum number of times the ant can cross its own path as it walks?

____________ **12** Find the number of ways to partition $S = \{1, 2, 3, \ldots, 2020\}$ into two disjoint sets A and B with $A \cup B = S$ so that if you choose an element a from A and an element b from B, $a + b$ is never a multiple of 20. A or B can be the empty set, and the order of A and B doesn't matter. In other words, the pair of sets (A, B) is indistinguishable from the pair of sets (B, A).

____________ **13** How many ordered pairs of positive integers (a, b) are there such that a right triangle with legs of length a, b has an area of p, where p is a prime number less than 100?

____________ **14** Mr. Schwartz has been hired to paint a row of 7 houses. Each house must be painted red, blue, or green. However, to make it aesthetically pleasing, he doesn't want any three consecutive houses to be the same color. Find the number of ways he can fulfill his task.

____________ **15** Bread draws a circle. He then selects four random distinct points on the circumference of the circle to form a convex quadrilateral. Kwu comes by and randomly chooses another 3 distinct points (none of which are the same as Bread's four points) on the circle to form a triangle. Find the probability that Kwu's triangle does not intersect Bread's quadrilateral, where two polygons intersect if they have at least one pair of sides intersecting.

____________ **16** What is the largest integer n with no repeated digits that is relatively prime to 6? Note that two numbers are considered relatively prime if they share no common factors besides 1.

____________ **17** $\triangle KWU$ is an equilateral triangle with side length 12. Point P lies on minor arc $\widehat{\mathrm{WU}}$ of the circumcircle of $\triangle KWU$. If $\overline{KP} = 13$, find the length of the altitude from P onto $\overline{WU}$.

____________ **18** Let w, x, y, z be integers from 0 to 3 inclusive. Find the number of ordered quadruples of (w, x, y, z) such that $5x^2 + 5y^2 + 5z^2 - 6wx - 6wy - 6wz$ is divisible by 4.

____________ **19** In a regular hexagon $ABCDEF$ of side length 8 and center K, points W and U are chosen on $\overline{AB}$ and $\overline{CD}$ respectively such that $\overline{KW} = 7$ and $\angle WKU = 120°$. Find the area of pentagon $WBCUK$.

____________ **20** Sam colors each tile in a 4 by 4 grid white or black. A coloring is called *rotationally symmetric* if the grid can be rotated 90, 180, or 270 degrees to achieve the same pattern. Two colorings are called *rotationally distinct* if neither can be rotated to match the other. How many rotationally distinct ways are there for Sam to color the grid such that the colorings are *not* rotationally symmetric?

____________ **21** Matthew Casertano and Fox Chyatte make a series of bets. In each bet, Matthew sets the stake (the amount he wins or loses) at half his current amount of money. He has an equal chance of winning and losing each bet. If he starts with \$256, find the probability that after 8 bets, he will have at least \$50.

____________ **22** Find the product of all positive real solutions to the equation $x^{-x} + x^{\frac{1}{x}} = \frac{2021}{2020}$

____________ **23** Let $ABCD$ be a cyclic quadrilateral so that $\overline{AC} \perp \overline{BD}$. Let E be the intersection of $\overline{AC}$ and $\overline{BD}$, and let F be the foot of the altitude from E to $\overline{AB}$. Let $\overline{EF}$ intersect $\overline{CD}$ at G, and let the foot of the perpendiculars from G to $\overline{AC}$ and $\overline{BD}$ be H, I respectively. If $\overline{AB} = \sqrt{5}, \overline{BC} = \sqrt{10}, \overline{CD} = 3\sqrt{5}, \overline{DA} = 2\sqrt{10}$, find the length of $\overline{HI}$.

______ **24** Nashan randomly chooses 6 positive integers a, b, c, d, e, f. Find the probability that $2^a + 2^b + 2^c + 2^d + 2^e + 2^f$ is divisible by 5.

______ **25** Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the sum of all positive integer solutions to

$$\left\lfloor \frac{n^3}{27} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor^3 = 10.$$

______ **26** Let $\triangle MBT$ be a triangle with $\overline{MB} = 4$ and $\overline{MT} = 7$. Furthermore, let circle ω be a circle with center O which is tangent to $\overline{MB}$ at B and $\overline{MT}$ at some point on segment $\overline{MT}$. Given $\overline{OM} = 6$ and ω intersects $\overline{BT}$ at $I \neq B$, find the length of $\overline{TI}$.

______ **27** The perfect square game is played as follows: player 1 says a positive integer, then player 2 says a strictly smaller positive integer, and so on. The game ends when someone says 1; that player wins if and only if the sum of all numbers said is a perfect square. What is the sum of all n such that, if player 1 starts by saying n, player 1 has a winning strategy? A winning strategy for player 1 is a rule player 1 can follow to win, regardless of what player 2 does. If player 1 wins, player 2 must lose, and vice versa. Both players play optimally.

______ **28** Consider the system of equations

$$a + 2b + 3c + \ldots + 26z = 2020$$

$$b + 2c + 3d + \ldots + 26a = 2019$$

$$\vdots$$

$$y + 2z + 3a + \ldots + 26x = 1996$$

$$z + 2a + 3b + \ldots + 26y = 1995$$

where each equation is a rearrangement of the first equation with the variables cycling and the coefficients staying in place. Find the value of

$$z + 2y + 3x + \cdots + 26a.$$

______ **29** The center of circle ω_1 of radius 6 lies on circle ω_2 of radius 6. The circles intersect at points K and W. Let point U lie on the major arc $\widehat{\mathrm{KW}}$ of ω_2, and point I be the center of the largest circle that can be inscribed in $\triangle KWU$. If $KI + WI = 11$, find $KI \cdot WI$.

____________ **30** Let the number of ways for a rook to return to its original square on a 4×4 chessboard in 8 moves if it starts on a corner be k. Find the number of positive integers that are divisors of k. A "move" counts as shifting the rook by a positive number of squares on the board along a row or column. Note that the rook may return back to its original square during an intermediate step within its 8-move path.

____________ **31** Consider the infinite sequence $\{a_i\}$ that extends the pattern

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

Formally, $a_i = i - T(i)$ for all $i \geq 1$, where $T(i)$ represents the largest triangular number less than i (triangle numbers are integers of the form $\frac{k(k+1)}{2}$ for some nonnegative integer k). Find the number of indices i such that $a_i = a_{i+2020}$.

____________ **32** Let the *square decomposition* of a number be defined as the sequence of numbers given by the following algorithm. Given a positive integer n, add the largest possible perfect square that is less than or equal to n to a sequence, and then subtract that number from n. Repeat as many times as necessary until your current n is 0. So for example, the square decomposition of 60 would be $49, 9, 1, 1$. Define the size of a square decomposition to be the number of numbers in the sequence. Say that the maximal size of a square decomposition of a number in the range $[1, 2020]$ is m. Find the largest number in the range $[1, 2020]$ that has a square decomposition of size m.

____________ **33** Circle ω_1 with center K of radius 4 and circle ω_2 of radius 6 intersect at points W and U. If the incenter of $\triangle KWU$ lies on circle ω_2, find the length of $\overline{WU}$. (Note: The incenter of a triangle is the intersection of the angle bisectors of the angles of the triangle)

____________ **34** Let a set S of n points be called *cool* if:

- All points lie in a plane
- No three points are collinear
- There exists a triangle with three distinct vertices in S such that the triangle contains another point in S strictly inside it

Define $g(S)$ for a cool set S to be the sum of the number of points strictly inside each triangle with three distinct vertices in S. Let $f(n)$ be the minimal possible value of $g(S)$ across all cool sets of size n. Find

$$f(4) + \cdots + f(2020) \pmod{1000}$$

______ **35** Tim has a multiset of positive integers. Let c_i be the number of occurrences of numbers that are *at least* i in the multiset. Let m be the maximum element of the multiset. Tim calls a multiset *spicy* if $c_1, \ldots, c_m$ is a sequence of strictly decreasing powers of 3. Tim calls the *hotness* of a spicy multiset the sum of its elements. Find the sum of the hotness of all spicy multisets that satisfy $c_1 = 3^{2020}$. Give your answer $\pmod{1000}$. (Note: a multiset is an unordered set of numbers that can have repeats)

______ **36** $ABCD$ is a rectangle $\overline{AB} = 5\sqrt{3}$, $\overline{AD} = 30$. Extend $\overline{BC}$ past C and construct point P on this extension such that $\angle APD = 60°$. Point H is on $\overline{AP}$ such that $\overline{DH} \perp \overline{AP}$. Find the length of $\overline{DH}$.

______ **37** Fuzzy likes isosceles trapezoids. He can choose lengths from $1, 2, \ldots, 8$, where he may choose any amount of each length. He takes a multiset of three integers from $1, \ldots, 8$. From this multiset, one length will become a base length, one will become a diagonal length, and one will become a leg length. He uses each element as either a diagonal, leg, or base length exactly once. Fuzzy is happy if he can use these lengths to make an isosceles trapezoid such that the undecided base has nonzero rational length. How many multiset choices can he make? (Multisets are unordered)

______ **38** Consider $\triangle ABC$ with circumcenter O and $\angle ABC$ obtuse. Construct A' as the reflection of A over O, and let P be the intersection of $\overline{A'B}$ and $\overline{AC}$. Let P' be the intersection of the circumcircle of (OPA) with $\overline{AB}$. Given that the circumdiameter of $\triangle ABC$ is 25, $\overline{AB} = 7$, and $\overline{BC} = 15$, find the length of PP'.

______ **39** Let $f(x) = \sqrt{4x^2 - 4x^4}$. Let A be the number of real numbers x that satisfy

$$f(f(f(\ldots f(x)\ldots))) = x,$$

where the function f is applied to x 2020 times. Compute $A \pmod{1000}$.

______ **40** Wu starts out with exactly one coin. Wu flips every coin he has *at once* after each year. For each heads he flips, Wu receives a coin, and for every tails he flips, Wu loses a coin. He will keep repeating this process each year until he has 0 coins, at which point he will stop. The probability that Wu will stop after exactly five years can be expressed as $\frac{a}{2^b}$, where a, b are positive integers such that a is odd. Find $a + b$.

____________ **41** What are the last two digits of $2^{3^{4^{\cdots^{2019}}}}$?

____________ **42** $\triangle ABC$ has side lengths $AB = 4$ and $AC = 9$. Angle bisector AD bisects angle A and intersects BC at D. Let k be the ratio $\frac{BD}{AB}$. Given that the length AD is an integer, find the sum of all possible k^2.

____________ **43** Let $\sigma_k(n)$ be the sum of the k^{th} powers of the divisors of n. For all $k \geq 2$ and all $n \geq 3$, we have that

$$\frac{\sigma_k(n)}{n^{k+2}}(2020n + 2019)^2 > m.$$

Find the largest possible value of m.

____________ **44** Let

$$a_n = \sum_{d|n} \frac{1}{2^{d+\frac{n}{d}}}.$$

In other words, a_n is the sum of $\frac{1}{2^{d+\frac{n}{d}}}$ over all divisors d of n. Find

$$\frac{\sum_{k=1}^{\infty} ka_k}{\sum_{k=1}^{\infty} a_k} = \frac{a_1 + 2a_2 + 3a_3 + \dots}{a_1 + a_2 + a_3 + \dots}$$

____________ **45** In the Flatland Congress there are senators who are on committees. Each senator is on at least one committee, and each committee has at least one senator. The rules for forming committees are as follows:

- For any pair of senators, there is exactly one committee which contains both senators.
- For any two committees, there is exactly one senator who is on both committees.
- There exist a set of four senators, no three of whom are all on the same committee.
- There exists a committee with exactly 6 senators.

If there are at least 25 senators in this Congress, compute the minimum possible number of senators s and minimum number of committees c in this Congress. Express your answer in the form (s, c).