MBMT Team Round — Leibniz

March 30, 2019

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **15** questions. You will have **45** minutes to complete the round. Later questions are worth more points; point values are notated next to the problem statement. (There are a total of 100 points.) Please write your answers in the simplest possible form.

DO NOT TURN THE QUESTION SHEET IN!
Use the official answer sheet.

You are highly encouraged to work with your teammates on the problems in order to solve them.

1 **[4]** At an interesting supermarket, the nth apple you purchase costs n dollars, while pears are 3 dollars each. Given that Layla has exactly enough money to purchase either k apples or $2k$ pears for $k > 0$, how much money does Layla have?

Proposed by Steven Qu

Solution. $\boxed{66}$

The cost of buying k apples is $1 + \cdots + k = \frac{k(k+1)}{2}$, and the cost of buying $2k$ pears is $6k$. Setting these equal, $k(k+1) = 12k$. By inspection or algebra, $k = 11$. Therefore, Layla has $\boxed{\$66}$. □

2 **[4]** Real numbers a, b, c are selected uniformly and independently at random between 0 and 1. What is the probability that $a \geq b \leq c$?

Proposed by Jacob Stavrianos

Solution. $\boxed{\dfrac{1}{3}}$

Define $P(x)$ to be the probability that x is the largest number, where $x \in \{a, b, c\}$. By symmetry, $P(a) = P(b) = P(c)$. Also, there's always a largest number, so $P(a) + P(b) + P(c) = 1$. Thus, $P(b) = \boxed{\frac{1}{3}}$. □

3 **[4]** For how many positive integers $1 \leq n \leq 10$ does there exist a prime p such that the sum of the digits of p is n?

Proposed by Daniel Zhu

Solution. $\boxed{7}$

1 doesn't work since 10^k is not prime for $k \geq 0$. $6, 9$ don't work because of the divisibility rule for 3. Below is a table for the n that work.

n	2	3	4	5	7	8	10
p	2	3	13	5	7	17	19

Thus there are $\boxed{7}$ such n. □

4 **[5]** What is the maximum number of intersection points between 2 circles and 2 triangles?

Proposed by Daniel Zhu

Solution. $\boxed{32}$

There are 2 intersections between the 2 circles. There are 6 intersections between the 2 triangles. Given any pair of a triangle and a circle, there are 6 intersections; there are 4 such pairs. So, the answer is $2 + 6 + (4 \cdot 6) = \boxed{32}$. □

5 **[5]** There are 50 dogs in the local animal shelter. Each dog is enemies with at least 2 other dogs. Steven wants to adopt as many dogs as possible, but he doesn't want to adopt any pair of enemies, since they will cause a ruckus. Considering all possible enemy networks among the dogs, find the maximum number of dogs that Steven can possibly adopt.

Proposed by Kevin A. Zhou

Solution. $\boxed{48}$

The best possible enemy network is the bipartite graph $K_{48,2}$, in other words, when there is a group of 48 dogs who are not enemies with each other, but all of them are enemies with the 2 dogs in the other group. Steven can simply adopt every dog in the group of size $\boxed{48}$. □

6 **[5]** How many ordered pairs of positive integers (x, y), where x is a perfect square and y is a perfect cube, satisfy $\operatorname{lcm}(x, y) = 81000000$?

Proposed by Daniel Zhu

Solution. $\boxed{72}$

First, factor $81000000 = 2^6 \cdot 3^4 \cdot 5^6$. Let $x = 2^a \cdot 3^b \cdot 5^c$ and $y = 2^d \cdot 3^e \cdot 5^f$. Note that $\max(a, d) = 6, \max(b, e) = 4, \max(c, f) = 6$. We can then list the possible pairs $(a, d), (b, e), (c, f)$:

(a, d)	(b, e)	(c, f)
$(6, 0)$	$(4, 0)$	$(6, 0)$
$(6, 3)$	$(4, 3)$	$(6, 3)$
$(6, 6)$		$(6, 6)$
$(4, 6)$		$(4, 6)$
$(2, 6)$		$(2, 6)$
$(0, 6)$		$(0, 6)$

Thus, there are $6 \cdot 2 \cdot 6 = \boxed{72}$ ordered pairs. □

7 **[6]** Unit circles a, b, c satisfy $d(a, b) = 1$, $d(b, c) = 2$, and $d(c, a) = 3$, where $d(x, y)$ is defined to be the minimum distance between any two points on circles x and y. Find the radius of the smallest circle entirely containing a, b, and c.

Proposed by Jacob Stavrianos

Solution. $\boxed{\dfrac{7}{2}}$

The centers of the circles form a 3-4-5 right triangle. Extend the hypotenuse to the ends of circles A, C to get a diameter of length 7, which corresponds to a radius of length $\boxed{\frac{7}{2}}$. □

______ **8** **[6]** The numbers 1 through 5 are written on a chalkboard. Every second, Sara erases two numbers a and b such that $a \geq b$ and writes $\sqrt{a^2 - b^2}$ on the board. Let M and m be the maximum and minimum possible values on the board when there is only one number left, respectively. Find the ordered pair (M, m).

Proposed by Daniel Zhu

Solution. $\boxed{(\sqrt{23}, \sqrt{3})}$

M is $\sqrt{25 - x^2}$, where x is made by applying Sara's process on $1, 2, 3, 4$ and made as small as possible. The best way to do this is $x = \sqrt{4^2 - 3^2 - 2^2 - 1^2} = \sqrt{2}$. Thus, $M = \sqrt{23}$.

To make m, we should first deal with $3, 4, 5$. If Sara does not end up with 0 after processing those numbers (by using $3^2 + 4^2 - 5^2$ or some other variant), then the remaining number on the chalkboard will be quite large. Therefore, Sara should do something like $m = \sqrt{2^2 - 1^2 - (5^2 - 4^2 - 3^2)} = \sqrt{3}$. □

______ **9** **[7]** N people stand in a line. Bella says, "There exists an assignment of nonnegative numbers to the N people so that the sum of all the numbers is 1 and the sum of any three consecutive people's numbers does not exceed 1/2019." If Bella is right, find the minimum value of N possible.

Proposed by Steven Qu

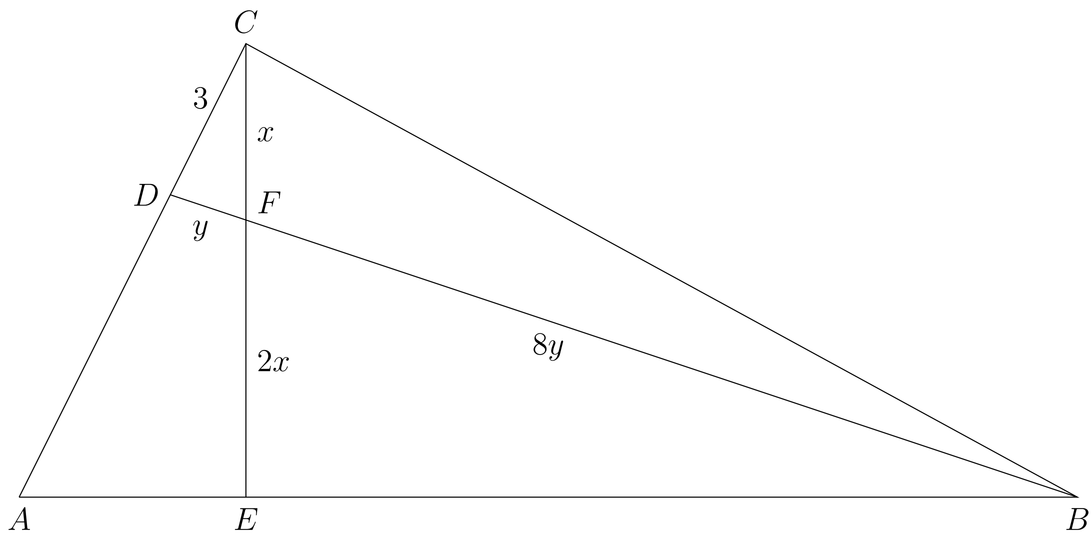
Solution. $\boxed{6055}$

Assign the numbers $\frac{1}{2019}, 0, 0, \frac{1}{2019}, 0, 0, \ldots \frac{1}{2019}$, with 2019 people assigned the number $\frac{1}{2019}$. This satisfies the problem, so $N = 6055$ is achievable. Now, suppose $N \leq 6054$. Split the N people into groups of size at most 3 by taking the first three as one group, the next three as a second group, and so on, until the last group, which may not have the full 3 people. This forms $\lceil \frac{N}{3} \rceil$ groups of size ≤ 3. By the Pigeonhole Principle, at least one of these groups will have a sum at least $\frac{1}{\lceil N/3 \rceil} \geq \frac{1}{2018} > \frac{1}{2019}$, a contradiction. Therefore, $N = 6055$ is the answer. □

______ **10** **[8]** In triangle $\triangle ABC$, D is on AC such that BD is an altitude, and E is on AB such that CE is an altitude. Let F be the intersection of BD and CE. If $EF = 2FC$, $BF = 8DF$, and $DC = 3$, then find the area of $\triangle CDF$.

Proposed by Kevin A. Zhou

Solution. $\boxed{\dfrac{3\sqrt{3}}{2}}$



We know that $\angle CDB \cong \angle CEB$ because they are right angles. Since they both inscribe the same arc, quadrilateral $CDEB$ is cyclic. Then, by Power of a Point, $x \cdot 2x = y \cdot 8y$, or $\frac{x}{y} = 2$. This means that $\triangle CDF$ is a 30-60-90 triangle, so $DF = \sqrt{3}$. Therefore, the area is $\boxed{\dfrac{3\sqrt{3}}{2}}$. □

11 **[8]** Consider nonnegative real numbers $a_1, \dots, a_6$ such that $a_1 + \cdots + a_6 = 20$. Find the minimum possible value of

$$\sqrt{a_1^2 + 1^2} + \sqrt{a_2^2 + 2^2} + \sqrt{a_3^2 + 3^2} + \sqrt{a_4^2 + 4^2} + \sqrt{a_5^2 + 5^2} + \sqrt{a_6^2 + 6^2}.$$

Proposed by Timothy Qian

Solution. $\boxed{29}$

Note that $\sqrt{a_i^2 + i^2}$ is the distance travelled from going up by a_i and right by i. Additionally, the entire sum is the distance of a path that goes from $(0, 0)$ to $(20, 21)$. The minimum distance of this path is $\sqrt{20^2 + 21^2} = \boxed{29}$. □

12 **[9]** Given two points A and B in the plane with $AB = 1$, define $f(C)$ to be the incenter of triangle ABC, if it exists. Find the area of the region of points $f(f(X))$ where X is arbitrary.

Proposed by Daniel Zhu

Solution. $\boxed{\pi/4 - 1/2}$

The set of such points X is the set of points with $\angle AXB > 135°$. This takes the form of two quarter-circle-minus-triangle shapes.

The radius of each circle is $1/\sqrt{2}$, so the two quarter circles have area $\pi/4$. Each triangle has area $1/4$, so the final answer is $\boxed{\pi/4 - 1/2}$. □

13 **[9]** Find an $a < 1000000$ so that both a and $101a$ are triangular numbers. (A triangular number is a number that can be written as $1 + 2 + \cdots + n$ for some $n \geq 1$.)

Note: There are multiple possible answers to this problem. You only need to find one.

Proposed by Daniel Zhu

Solution. $\boxed{4095 \text{ or } 6105}$

Let $101a = \frac{1}{2}n(n+1)$. One of n or $n+1$ must be divisible by 101. Since we only need to find one solution, we guess that there exists an answer with $n = 101k$, which turns out to be true. If this were not the case, we could handle the $n = 101k - 1$ case similarly.

We have $a = \frac{1}{2}k(101k + 1)$, so $m(m + 1) = k(101k + 1)$ for some m. Therefore $m^2 + m - k(101k + 1) = 0$. By the quadratic formula, for an integer solution to exist, the discriminant $1^2 + 4k(101k + 1)$ must be a perfect square. Set $r = 10$. Then we need $4r^2k^2 + 4k^2 + 4k + 1$ to be a square; call this p^2. Note that $(2rk)^2 = 4r^2k^2$, so let $q = 2rk$.

We have $p^2 - q^2 = 4k^2 + 4k + 1$. This is relatively small compared to p^2 and q^2, so we will try different values for $d = p - q$. If $d = 1$, then $2q + 1 = 4k^2 + 4k + 1 \implies 4k^2 - 36k = 0$. So $k = 9$ is a potential solution.

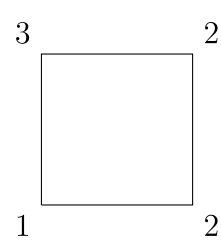
Under this solution $p = 2rk + 1 = 181$. So by returning to the quadratic formula, $m = \frac{1}{2}(-1 \pm 181) = 90, -91$. So $m = 90$. So $a = \frac{1}{2}m(m + 1) = \frac{1}{2}8190 = \boxed{4095}$. $\square$

14 **[10]** Leptina and Zandar play a game. At the four corners of a square, the numbers 1, 2, 3, and 4 are written in clockwise order. On Leptina's turn, she must swap a pair of adjacent numbers. On Zandar's turn, he must choose two adjacent numbers a and b with $a \geq b$ and replace a with $a - b$. Zandar wants to reduce the sum of the numbers at the four corners of the square to 2 in as few turns as possible, and Leptina wants to delay this as long as possible. If Leptina goes first and both players play optimally, find the minimum number of turns Zandar can take after which Zandar is guaranteed to have reduced the sum of the numbers to 2.

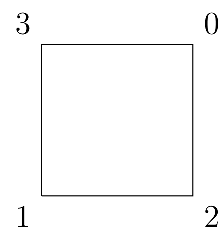
Proposed by Haydn Gwyn

Solution. $\boxed{5}$

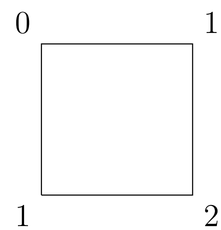
Turn 1: No matter what pair of numbers Leptina swaps, 2 and 4 will be next to each other. As such, Zandar replaces the 4 with a 2. Here is the guaranteed position after turn 1 (ignoring reflections and rotations):



Turn 2: After Leptina's swap, Zandar subtracts 2 from the number on the opposite corner of the 1. He can guaranteedly do this because one of the two numbers adjacent to the number he replaces must be a 2.

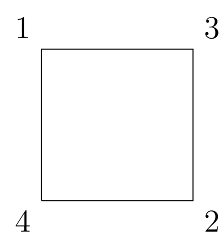


Turn 3: The two remaining numbers that are at least 2 are on opposite corners. Therefore, they will be next to each other after Leptina's swap. Zandar can thus subtract by 2 again, bringing the sum of the numbers to 4.



Turns 4 and 5: Since there is only one 2 remaining, each of Zandar's moves will decrease the sum by 1. He can replace the 2 with a 1, and finally a 1 with a 0, to achieve his goal in 5 moves.

Proving fewer than 5 moves is impossible: Let's suppose that Zandar can reduce the sum in 4 moves or fewer. Then, the average decrease in the sum of the numbers per turn is at least $\frac{10-2}{4} = 2$. This implies that Zandar must subtract 2 every turn, since Leptina can prevent Zandar from subtracting by 3 on the first turn by swapping the 1 and 4. This is since either the 3 or 4 must become smaller, and thus Zandar can never subtract by 3 in the future.



Zandar cannot subtract by 2 every turn however. At the end of the game, there must be two odd numbers since subtracting 2 does not change parity. Additionally, the number 2 must exist. Thus, the sum is at least $1+1+2=4$, a contradiction. Therefore, Zandar requires $\boxed{5}$ turns. $\square$

____________ **15** **[10]** There exist polynomials P, Q and real numbers $c_0, c_1, c_2, \ldots, c_{10}$ so that the three polynomials

$$P, \quad Q, \text{ and } \quad c_0P^{10} + c_1P^9Q + c_2P^8Q^2 + \cdots + c_{10}Q^{10}$$

are all polynomials of degree 2019. Suppose that $c_0 = 1, c_1 = -7, c_2 = 22$. Find all possible values of c_{10}.

Note: The answer(s) are rational numbers. It suffices to give the prime factorization(s) of the numerator(s) and denominator(s).

Proposed by Daniel Zhu

Solution. $\boxed{\dfrac{2^9}{3^9}, \dfrac{2 \cdot 11^9}{3^9 \cdot 5^{10}}}$

Notice that we can factor

$$c_0P^{10} + c_1P^9Q + c_2P^8Q^2 + \cdots + c_{10}Q^{10} = \prod_i (P - z_iQ)$$

for some complex numbers z_i. Since $P - z_iQ$ can have a degree less than 2019 for only one value of z_i, then we must have

$$c_0x^{10} + c_1x^9 + \cdots + c_{10} = (x-a)^9(x-b)$$

for some a and b. The conditions yield $9a + b = 7$ and $36a^2 + 9ab = 22$, and solving this yields $(a, b) = (2/3, 1), (11/15, 2/5)$. These each yield a solution for $c_{10} = a^9b$. $\square$