MBMT Number Theory Round — Leibniz

March 30, 2019

Full Name ____

Team Number _________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 On Jupiter, a day is 10 hours long. Jim is a strange animal on Jupiter who eats a rock every 3 hours. Exactly at midnight, Jim eats a rock. How many hours will pass before he eats again at midnight?

Proposed by Ambrose Yang

Solution. $\boxed{30}$

Listing out the times he eats, we have 0, 3, 6, 9, 2, 5, 8, 1, 4, 7, 0. Thus, $\boxed{30}$ hours pass between two consecutive times he eats a rock at midnight. □

2 What is the remainder when $2017^2 + 2018^2 + 2019^2 + 2020^2 + 2021^2$ is divided by 5?

Proposed by Jacob Stavrianos

Solution. $\boxed{0}$

Since 2017, 2018, 2019, 2020, 2021 are consecutive numbers, they contain each value mod 5 exactly once. Thus, we can instead compute $0^2 + 1^2 + 2^2 + 3^2 + 4^2$ mod 5, which we evaluate to be $\boxed{0}$. □

3 We call a year *summable* if there exists some day during the year such that the sum of the month and the day equals the last two digits of the year. Find the first year after 2018 that is not summable.

Proposed by Ambrose Yang

Solution. $\boxed{2044}$

We note that the greatest sum of the month and day of some day during the year is $12 + 31 = 43$ and that all integers between 19 and 43 can be expressed as the sum of a month value and a day value. Thus, $\boxed{2044}$ is the first year after 2018 that is not *summable*. □

4 Find the largest multiple of 4 that has fewer than six positive integer factors.

Proposed by Haydn Gwyn

Solution. $\boxed{16}$

We first claim that the answer is a power of two. If not, then it would have some other prime factor p. The means that it has factors 1, 2, 4, p, $2p$, and $4p$, which is already six factors. As such, we simply look at the powers of two: 4, 8, 16, 32, etc. We note that 4 has three factors, 8 has four factors, 16 has five factors, and 32 has six factors. Also, every power of two above 32 has at least six factors (1, 2, 4, 8, 16, 32), so our answer is $\boxed{16}$. □

5 Define a real number x to be *imbalanced* if its decimal expansion is infinite and, in the decimal expansion of x, all but a finite number of digits have the same nonzero value. For instance, 0.123 is not imbalanced since its decimal expansion is finite. What is the smallest n such that, for all real numbers x, at least one of $x, 2x, \ldots, nx$ is **not** imbalanced?

Proposed by Jacob Stavrianos

Solution. $\boxed{9}$

We note that, if x is imbalanced, then there exists some $m \in \mathbb{Z}$ such that all digits after the 10^m digit have the same value k. Thus, we can express x as $x_t + \frac{k}{9}10^m$, where x_t is x truncated after the 10^m digit.

From here, we get $nx = nx_t + \frac{nk}{9}10^m$. If $\frac{nk}{9}$ is an integer, than nx is finitely terminating, which means it can't be imbalanced. Thus, we get $n = \boxed{9}$ is an upper bound; we get an exact bound by noting that $x = \frac{1}{9}$ requires $n = 9$. □

6 The number $\frac{1}{2310}$ can be written in the form $\frac{1}{a} - \frac{1}{b}$, where a and b are positive integers and $a + b$ is as small as possible. Find $b - a$.

Proposed by Haydn Gwyn

Solution. $\boxed{5}$

From the equation $\frac{1}{2310} = \frac{1}{a} - \frac{1}{b}$, we get $(a - 2310)(b + 2310) = -2310^2$ through some algebra and Simon's Favorite Factoring Trick. Since we want to minimize $a + b$, we also want to minimize $(a - 2310) + (b + 2310)$. Moreover, these 2 terms multiply to some fixed number. Therefore, we want to find 2 numbers j, k with very close magnitudes such that $jk = -2310^2$. The best way to do this is set $j = 2310, k = 2310$, and then multiply j by $\frac{21}{22}$ and multiply k by $\frac{22}{21}$. Now, we have $j = 2205, k = 2420$, so $a = 105, b = 110$. Therefore, $b - a = \boxed{5}$. □

7 Find the number of positive integers $k \leq 100$ such that $6^k + 2 \cdot 5^k - 1$ is divisible by 31.

Proposed by Timothy Qian

Solution. $\boxed{17}$

It is not hard to see that $5 \equiv 6^2 \pmod{31}$, so letting $m = 6^k$, the equation becomes $2n^2 + n - 1 \equiv 0$, which reduces to $n \equiv -1, 1/2$. Since $6^6 \equiv 1 \pmod{31}$, so we only check the mods less than 6, and we get that $x \equiv 3 \mod 6$ all work, so we simply count them to get $\boxed{17}$. □

8 Let a *pegboard* be a (possibly rotated) infinite square grid of points with side length 1 in the plane. Call two pegboards *equivalent* if one is a translation of another. Let the *gap* between two pegboards be the minimum distance between a point in each pegboard. Find the maximum gap between two non-equivalent pegboards.

Proposed by Daniel Zhu

Solution. $\boxed{1/\sqrt{10}}$

WLOG let one pegboard be the lattice points and let the other be generated by directions (x, y) and $(-y, x)$ where $x^2 + y^2 = 1$ (WLOG $x, y > 0$). We split the analysis into two cases.

If x and y are both rational, then notice that they must have the same denominator when written in lowest terms; call this n. We claim that mod 1, the second pegboard forms a square grid with side length $1/\sqrt{n}$. It is then clear that the maximum gap is $1/\sqrt{2n}$. Since all rational pegboards come from Pythagorean triples, this means that maximum is $\boxed{1/\sqrt{10}}$.

Let $x = a/n$ and $y = b/n$. Notice that mod n, $-y/x \cdot y = x$, so the set of points mod 1 generated by (x, y) is the same as those generated by $(-y, x)$. Therefore there is a lattice with n points in each unit square, and since it is invariant under rotation by 90 degrees, it must be a square lattice.

If one of x and y is irrational, we claim that the elements of the second pegboard are dense mod 1. To see this, note that by Pigeonhole there exists an n with the fractional part of (nx, ny) arbitrarily small, allowing (nx, ny) and $(-ny, nx)$ to form an arbitrarily dense cover of the unit square. Thus the gap can be arbitrarily small. □