MBMT Leibniz Guts Round — Set 1

March 30, 2019

____________ **1** **[3]** Find the units digit of $3^{1^{3^{3^{7}}}}$.

____________ **2** **[3]** A standard deck of cards contains 13 cards of each suit (clubs, diamonds, hearts, and spades). After drawing 51 cards from a randomly ordered deck, what is the probability that you have drawn an odd number of clubs?

____________ **3** **[3]** Square $ABCD$ with side length 1 is rolled into a cylinder by attaching side AD to side BC. What is the volume of that cylinder?

____________ **4** **[3]** Haydn is selling pies to Grace. He has 4 pumpkin pies, 3 apple pies, and 1 blueberry pie. If Grace wants 3 pies, how many different pie orders can she have?

____________ **5** **[3]** Kevin has written 5 MBMT questions. The shortest question is 5 words long, and every other question has exactly twice as many words as a different question. Given that no two questions have the same number of words, how many words long is the longest question?

MBMT Leibniz Guts Round — Set 2

March 30, 2019

____________ **6** **[4]** Alex has 100 Bluffy Funnies in some order, which he wants to sort in order of height. They're already *almost* in order: each Bluffy Funny is at most 1 spot off from where it should be. Alex can only swap pairs of adjacent Bluffy Funnies. What is the maximum possible number of swaps necessary for Alex to sort them?

____________ **7** **[4]** Let $s(n)$ be the sum of the digits of n. Let $g(n)$ be the number of times s must be applied to n until it has only 1 digit. Find the smallest n greater than 2019 such that $g(n) \neq g(n+1)$.

____________ **8** **[4]** In the Montgomery Blair Meterology Tournament, individuals are ranked (without ties) in ten categories. Their overall score is their average rank, and the person with the lowest overall score wins.

Alice, one of the 2019 contestants, is secretly told that her score is S. Based on this information, she deduces that she has won first place, without tying with anyone. What is the maximum possible value of S?

____________ **9** **[4]** Let A and B be opposite vertices on a cube with side length 1, and let X be a point on that cube. Given that the distance along the surface of the cube from A to X is 1, find the maximum possible distance along the surface of the cube from B to X.

____________ **10** **[4]** Given the following system of equations where x, y, z are nonzero, find $x^2 + y^2 + z^2$.

$$x + 2y = xy$$
$$3y + z = yz$$
$$3x + 2z = xz$$

MBMT Leibniz Guts Round — Set 3

March 30, 2019

____________ **11** **[5]** How many triples of nonnegative integers (x, y, z) satisfy the equation $6x + 10y + 15z = 300$?

____________ **12** **[5]** Point P inside square $ABCD$ is connected to each corner of the square, splitting the square into four triangles. If three of these triangles have area 25, 25, and 15, what are all the possible values for the area of the fourth triangle?

____________ **13** **[5]** Tse and Cho are playing a game. Cho chooses a number $x \in [0, 1]$ uniformly at random, and Tse guesses the value of $x(1 - x)$. Tse wins if his guess is at most $\frac{1}{50}$ away from the correct value. Given that Tse plays optimally, what is the probability that Tse wins?

____________ **14** **[5]** Hungryman starts at the tile labeled "S". On each move, he moves 1 unit horizontally or vertically and eats the tile he arrives at. He cannot move to a tile he already ate, and he stops when the sum of the numbers on all eaten tiles is a multiple of nine. Find the minimum number of tiles that Hungryman eats.

S	7	9	16	18
25	27	36	45	52
54	63	70	72	81
88	90	99	108	115
117	124	126	133	135

____________ **15** **[5]** A semicircle of radius 1 has line ℓ along its base and is tangent to line m. Let r be the radius of the largest circle tangent to ℓ, m, and the semicircle. As the point of tangency on the semicircle varies, the range of possible values of r is the interval $[a, b]$. Find $b - a$.

MBMT Leibniz Guts Round — Set 4

March 30, 2019

____________ **16** **[7]** Anson, Billiam, and Connor are looking at a 3D figure. The figure is made of unit cubes and is sitting on the ground. No cubes are floating; in other words, each unit cube must either have another unit cube or the ground directly under it. Anson looks from the left side and says, "I see a 5×5 square." Billiam looks from the front and says the same thing. Connor looks from the top and says the same thing. Find the absolute difference between the minimum and maximum volume of the figure.

____________ **17** **[7]** The repeating decimal $0.\overline{MBMT}$ is equal to $\frac{p}{q}$, where p and q are relatively prime positive integers, and M, B, T are distinct digits. Find the minimum value of q.

____________ **18** **[7]** Annie, Bob, and Claire have a bag containing the numbers $1, 2, 3, \ldots, 9$. Annie randomly chooses three numbers without replacement, then Bob chooses, then Claire gets the remaining three numbers. Find the probability that everyone is holding an arithmetic sequence. (Order does not matter, so 123, 213, and 321 all count as arithmetic sequences.)

____________ **19** **[7]** Consider a set S of positive integers. Define the operation $f(S)$ to be the smallest integer $n > 1$ such that the base 2^k representation of n consists only of ones and zeros for all $k \in S$. Find the size of the largest set S such that $f(S) < 2^{2019}$.

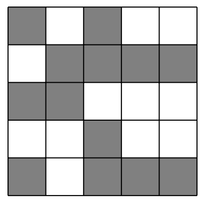
____________ **20** **[7]** Find the largest solution to the equation

$$2019(x^{2019x^{2019}-2019^2+2019})^{2019} = 2019^{x^{2019}+1}.$$

MBMT Leibniz Guts Round — Set 5

March 30, 2019

____________ **21** **[9]** Steven is concerned about his artistic abilities. To make himself feel better, he creates a 100×100 square grid and randomly paints each square either white or black, each with probability $\frac{1}{2}$. Then, he divides the white squares into connected components, groups of white squares that are connected to each other, possibly using corners. (For example, there are three connected components in the following diagram.) What is the expected number of connected components with 1 square, to the nearest integer?



____________ **22** **[9]** Let x be chosen uniformly at random from $[0, 1]$. Let n be the smallest **positive** integer such that $3^n x$ is at most $\frac{1}{4}$ away from an integer. What is the expected value of n?

____________ **23** **[9]** Let A and B be two points in the plane with $AB = 1$. Let ℓ be a variable line through A. Let ℓ' be a line through B perpendicular to ℓ. Let X be on ℓ and Y be on ℓ' with $AX = BY = 1$. Find the length of the locus of the midpoint of XY.

____________ **24** **[9]** Each of the numbers a_i, where $1 \le i \le n$, is either -1 or 1. Also,

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_{n-3}a_{n-2}a_{n-1}a_n + a_{n-2}a_{n-1}a_na_1 + a_{n-1}a_na_1a_2 + a_na_1a_2a_3 = 0.$$

Find the number of possible values for n between 4 and 100, inclusive.

____________ **25** **[9]** Let S be the set of positive integers less than 3^{2019} that have only zeros and ones in their base 3 representation. Find the sum of the squares of the elements of S. Express your answer in the form $a^b(c^d - 1)(e^f - 1)$, where a, b, c, d, e, f are positive integers and a, c, e are not perfect powers.

MBMT Leibniz Guts Round — Set 6

March 30, 2019

This round is an estimation round. No one is expected to get an exact answer to any of these questions, but unlike other rounds, you will get points for being close. In the interest of transparency, the formulas for determining the number of points you will receive are located on the answer sheet, but they aren't very important when solving these problems.

To receive points, your answers should be positive and in decimal notation. For example, 10.55 is allowed, but not -3.2 or $\frac{2\pi}{3}$.

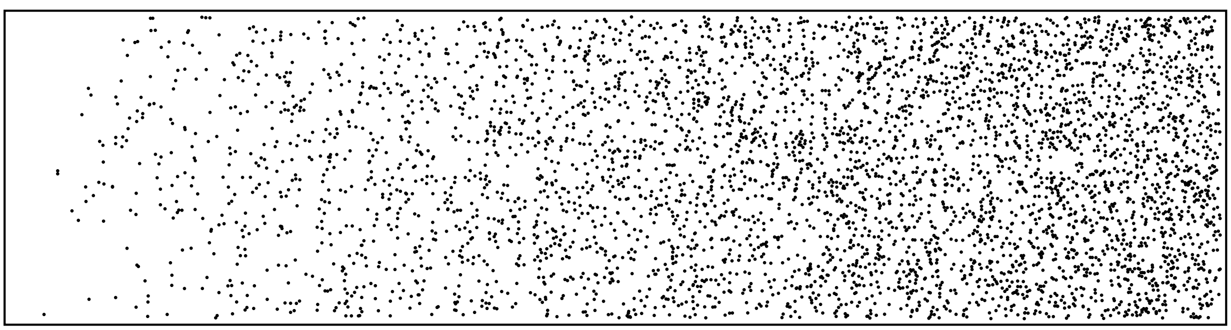
____________ **26** **[12]** What is the sum over all MBMT volunteers of the number of times that volunteer has attended MBMT (as a contestant or as a volunteer, including this year)?

Last year there were 47 volunteers; this is the fifth MBMT.

____________ **27** **[12]** William is sharing a chocolate bar with Naveen and Kevin. He first randomly picks a point along the bar and splits the bar at that point. He then takes the smaller piece, randomly picks a point along it, splits the piece at that point, and gives the smaller resulting piece to Kevin. Estimate the probability that Kevin gets less than 10% of the entire chocolate bar.

____________ **28** **[12]** Let x be the positive solution to the equation $x^{x^{x^x}} = 1.1$. Estimate $\frac{1}{x-1}$.

____________ **29** **[12]** Estimate the number of dots in the following box:



It may be useful to know that this image was produced by plotting $(4\sqrt{x}, y)$ some number of times, where x, y are random numbers chosen uniformly randomly and independently from the interval $[0, 1]$.

____________ **30** **[12]** For a positive integer n, let $f(n)$ be the smallest prime greater than or equal to n. Estimate

$$(f(1) - 1) + (f(2) - 2) + (f(3) - 3) + \cdots + (f(10000) - 10000).$$

For $26 \leq i \leq 30$, let E_i be your team's answer to problem i and let A_i be the actual answer to problem i. Your score S_i for problem i is given by

$$
\begin{aligned}
S_{26} &= \max(0, 12 - |E_{26} - A_{26}|/5) \\
S_{27} &= \max(0, 12 - 100|E_{27} - A_{27}|) \\
S_{28} &= \max(0, 12 - 5|E_{28} - A_{28}|)) \\
S_{29} &= 12 \max\left(0, 1 - 3\frac{|E_{29} - A_{29}|}{A_{29}}\right) \\
S_{30} &= \max(0, 12 - |E_{30} - A_{30}|/2000)
\end{aligned}
$$