MBMT Geometry Round — Leibniz

March 30, 2019

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

____________ **1** Let $ABCDEF$ be a regular hexagon. Given that $AD = 5$, find AB.

Proposed by Daniel Zhu

Solution. $\boxed{2.5}$

Dissect the hexagon into six equilateral triangles. Then it is clear that AD is twice the side length, so $AB = \boxed{\frac{5}{2}}$. □

____________ **2** Circles A, B, and C are all externally tangent, with radii 1, 10, and 100, respectively. What is the radius of the smallest circle entirely containing all three circles?

Proposed by Jacob Stavrianos

Solution. $\boxed{110}$

Consider the line segment that goes through the centers of B, C, and stops at the ends of the circles. This segment has length $200 + 20 = 220$, and since it must be entirely contained within the circle, the answer is at least 110. Since it is the diameter of a circle that contains all three circles, the radius is $\boxed{110}$. □

____________ **3** The hour hand of a clock is 6 inches long, and the minute hand is 10 inches long. Find the area of the region swept out by the hands from 8:45AM to 9:15AM of a single day, in square inches.

Proposed by Daniel Zhu

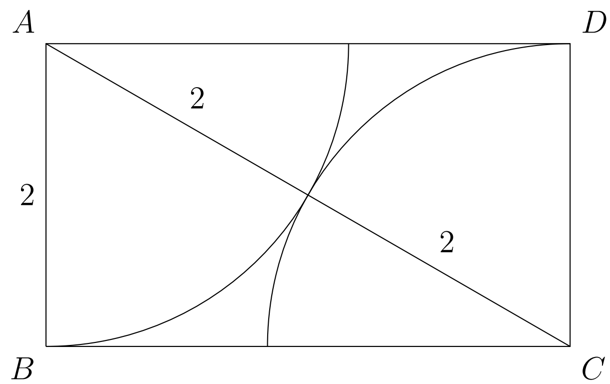
Solution. $\boxed{\dfrac{203\pi}{4}}$

The minute hand travels 180 degrees, so it sweeps out $\frac{100\pi}{2}$. The hour hand travels 15 degrees, but it overlaps with the minute hand half the time. So, it sweeps out $\frac{36\pi}{48}$. The total is $\boxed{\frac{203\pi}{4}}$. □

____________ **4** In rectangle $ABCD$, $AB = 2$ and $AD > AB$. Two quarter circles are drawn inside of $ABCD$ with centers at A and C that pass through B and D, respectively. If these two quarter circles are tangent, find the area inside of $ABCD$ that is outside both of the quarter circles.

Proposed by Haydn Gwyn

Solution. $\boxed{4\sqrt{3} - 2\pi}$



Draw segment AC. Observe that, by symmetry, the point of tangency of the two quarter circles is the midpoint of this segment, so the length of AC is $2+2=4$ (note that the radius of both circles is 2, as AB and CD are radii of the quarter circles). Then we have $AC=\sqrt{4^2-2^2}=2\sqrt{3}$. The area of the rectangle is then $2\cdot 2\sqrt{3}=4\sqrt{3}$, and the area of both of the quarter circles is $\frac{2}{4}\cdot\pi(2^2)=2\pi$. Thus, the area inside of the rectangle but outside of both quarter circles is $4\sqrt{3}-2\pi$. □

______ **5** Find the area of a triangle with side lengths $\sqrt{2}$, $\sqrt{58}$, and $2\sqrt{17}$.

Proposed by Olivia Fan

Solution. $\boxed{5}$

Impose the triangle onto a coordinate system with coordinates (0, 0), (3, 7), (2, 6). Apply shoelace. □

______ **6** Triangle ABC is equilateral. A circle passes through A and is tangent to side BC. It intersects sides AB and AC again at E and F, respectively. If $AE=10$ and $AF=11$, find AB.

Proposed by Daniel Zhu

Solution. $\boxed{7+\dfrac{2\sqrt{111}}{3}}$

Let x be the side-length. By using Power of a Point on B and C we get

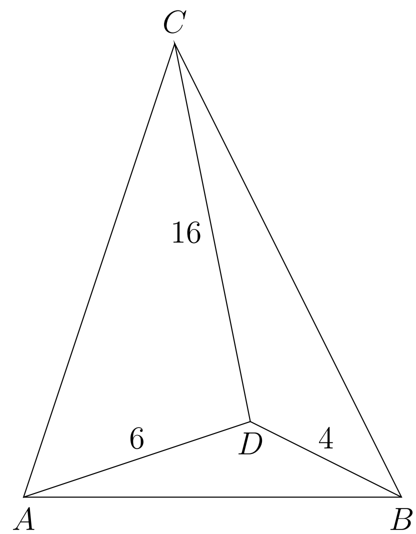
$$x=\sqrt{x(x-10)}+\sqrt{x(x-11)}.$$

The rest is computation. □

______ **7** Triangle ABC has area 80. Point D is in the **interior** of $\triangle ABC$ such that $AD=6, BD=4, CD=16$, and the area of $\triangle ADC=48$. Determine the area of $\triangle ADB$.

Proposed by Kevin A. Zhou

Solution. $\boxed{\dfrac{576}{73}}$



We repeatedly apply an area formula for a triangle. $[ADC] = 48 = \frac{1}{2}(6)(16)\sin\angle ADC$, so $\angle ADC = 90°$. Let $\angle ADB = \alpha$. Since $\angle BDC + \alpha = 270°$, then $\sin\angle BDC = \sin(270° - \alpha) = \sin(-90° - \alpha) = -\sin(90° + \alpha) = -\cos\alpha$.

$[BDC] = 32\sin\angle BDC = -32\cos\alpha$, and $[ADB] = 12\sin\alpha$. Since $[BDC] + [ADB] = 32$, we get the following equation, which we solve:

$$\begin{aligned} 12\sin\alpha - 32\cos\alpha &= 32 \\ 3\sin\alpha - 8\cos\alpha &= 8 \\ 3\sin\alpha - 8\sqrt{1-\sin^2\alpha} &= 8 \\ 8\sqrt{1-\sin^2\alpha} &= 3\sin\alpha - 8 \\ 64 - 64\sin^2\alpha &= 9\sin^2\alpha - 48\sin\alpha + 64 \\ 73\sin^2\alpha &= 48\sin\alpha \\ \sin\angle ADB &= \frac{48}{73} \end{aligned}$$

Since $AD = 6$ and $BD = 4$, the area of $\triangle ADB = \frac{576}{73}$. □

8 Given two points A and B in the plane with $AB = 1$, define $f(C)$ to be the circumcenter of triangle ABC, if it exists. Find the number of points X so that $f^{2019}(X) = X$.

Proposed by Daniel Zhu

Solution. $\boxed{2^{2019} - 2}$

Clearly X must lie on the perpendicular bisector of AB; call this line ℓ. Let P be a "point" that is arbitrarily far in one direction of ℓ. Consider the bijection $\varphi\colon \ell \to \mathbb{R}/\pi\mathbb{Z} \setminus \{0\}$ where $\varphi(T) = \angle ATP$.

Observe that under φ, f corresponds to multiplication by 2, so we want the number of solutions to $(2^{2019} - 1)x \equiv 0 \pmod{\pi}$, excluding 0, which is $\boxed{2^{2019} - 2}$. □