MBMT Counting and Probability Round — Leibniz

March 30, 2019

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 An ant is at one corner of a cube of side length 1 and can move only along the edges of the cube. How many paths of length 3 can the ant take to the opposite corner of the cube?

Proposed by Jacob Stavrianos

Solution. $\boxed{6}$

The ant has 3 choices for the first vertex it goes to. Then, the ant and its destination are on the same face of the cube. There are 2 ways for the ant to move along that face to its destination. Thus, $2 \cdot 3 = \boxed{6}$ ways. □

2 Felix the Frog is in the middle of an endless staircase. On every hop, he can either hop 9 steps down or 5 steps up. Felix hops 100 times. At how many possible locations can Felix end his hopping route?

Proposed by Kevin A. Zhou

Solution. $\boxed{101}$

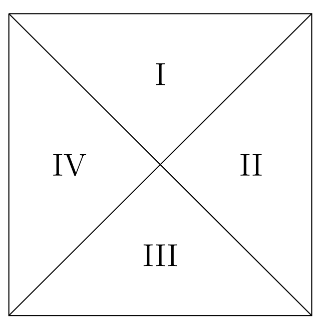
Felix can hop downward a number from 0 to 100 times. This then determines which step of the staircase he will be at. Therefore, there are $\boxed{101}$ possible locations where Felix ends his hopping route. □

3 Two points are randomly selected inside a rectangle. What is the probability that the segment connecting these two points crosses at least one of the rectangle's diagonals?

Proposed by Steven Qu

Solution. $\boxed{\dfrac{3}{4}}$

Define the following regions:



Notice that the segment crosses no diagonals if and only if the two points lie in the same region (I, II, III, or IV). By symmetry, we find that the probability of this *not* happening is $1 - \frac{1}{4} = \boxed{\frac{3}{4}}$. □

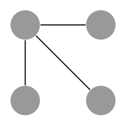
4 In a class of 4 students, everyone is friends with each other. (No one is friends with themselves, so everyone has 3 friends.) How many ways are there to break at least one of these friendships so that everyone still has an odd number of friends?

Proposed by Kevin Qian and Kevin A. Zhou

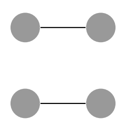
Solution. $\boxed{7}$

At least 1 friendship is broken, so someone only has 1 friend. Additionally, there cannot be more than two people with 3 friends, because then everyone would be friends with those 2+ people, and thus no one has only 1 friend. So, we have two cases.

Case 1. 1 person has 3 friends. There are 4 ways to pick the person with 3 friends. Then, all other friendships are determined:



Case 2. Nobody has 3 friends. Here, everyone has 1 friend. Therefore, there are 2 disjoint couples. There are $\binom{4}{2}$ ways to choose 1 couple, and then the other couple is determined. But, this overcounts by a factor of 2, because we could have chosen the other couple to get the same friendship network. So, there are 3 ways in this case.



In total, there are $\boxed{7}$ ways. □

5 Given a regular tetrahedron, how many ways are there to color two edges red, two edges green, and two edges blue? Rotations and reflections of a configuration are considered the same configuration.

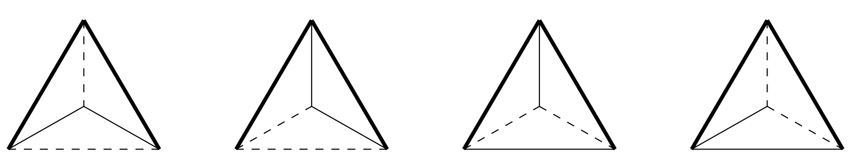
Note: A regular tetrahedron is a triangular pyramid with all faces equilateral triangles.

Proposed by Daniel Zhu

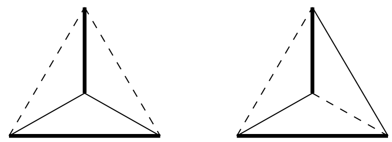
Solution. $\boxed{6}$

We will consider the tetrahedron as a 2D graph, since 3D is hard to visualize. Also, in the following diagrams, red edges are thick, green edges are normal, and blue edges are dashed.

Case 1. Red edges share a vertex. There are 4 ways.



Case 2. The red edges do not share a vertex. There are 2 ways.



□

6 Mr. Anderson rolls n fair six-sided dice, where $n \in \{1,2,3,4,5,6\}$ is chosen uniformly at random. Given that the sum of the dice rolls is 6, find the probability that $n = 3$.

Proposed by Kevin A. Zhou

Solution. $\boxed{\frac{2160}{16807}}$

If there are k dice, then the amount of ways for the dice rolls to sum to 6 is $\binom{5}{k-1}$. This is because we can consider 6 dots, and place dividers in the 5 blank spaces between them. Therefore, the probability that k dice rolls to sum 6 is $\frac{\binom{5}{k-1}}{6^k}$. The total probability is therefore $\sum_{k=1}^{6} \frac{\binom{5}{k-1}}{6^k} = \frac{1}{6}(1+\frac{1}{6})^5 = \frac{7^5}{6^6}$, by the binomial theorem. The probability that 3 dice rolls sum to 6 is $\frac{10}{6^3}$. So, our answer is

$$\frac{\frac{10}{6^3}}{\frac{7^5}{6^6}} = \frac{10 \cdot 6^3}{7^5} = \boxed{\frac{2160}{16807}}.$$

□

7 Steven starts with the number 1. Then, he repeats the following procedure N times: if he has the number n, he adds a random integer from 1 to $\gcd(n, 4)$, inclusive, to n. If $N = 2019^{2019^{2019}}$, find the closest integer to $100p$, where p is the probability that Steven's final number is divisible by 4.

Proposed by Daniel Zhu

Solution. $\boxed{44}$

There are four states:

- State A: Steven's number is $0 \pmod 4$. $\frac{1}{4}$ chance of going to each of A, B, C, D
- State B: Steven's number is $1 \pmod 4$. Certain to go to C
- State C: Steven's number is $2 \pmod 4$. $\frac{1}{2}$ chance of going to each of A, D
- State D: Steven's number is $3 \pmod 4$. Go to A

Since Steven takes a near infinite number of steps ($2019^{2019^{2019}}$), the probabilities can be approximated by the steady state. We can solve for this by supposing that a, b, c, and d are the probabilities of arriving at each state in the steady state. Then

$$\begin{aligned} a &= a/4 + c/2 + d \\ b &= a/4 \\ c &= a/4 + b \\ d &= a/4 + c/2 \end{aligned}$$

So $c = a/2 \implies d = a/2$. Since $a + b + c + d = 1$, we have

$$a = \frac{1}{1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{2}} = \frac{4}{9}.$$

Thus p is extremely close to $\frac{4}{9}$, so the closet integer to $100p$ is $\boxed{44}$. □

8 There are 4 traffic lights placed uniformly at random on a 4-mile road. The traffic lights, all in sync, follow a 1-minute loop where they are red for 1 minute and then momentarily flash green so that only cars already stopped at the light can go on. A car arrives at one end of the road just as the lights are flashing green. If the car, when not stopped, always travels at 1 mile per minute, what is the expected number of minutes it takes for the car to traverse the road?

Proposed by Daniel Zhu

Solution. $\boxed{\frac{1013}{160}}$

Let $d_1, d_2, \ldots, d_5$ be the distances between the lights in miles. We want to find the expected value of $\sum_{i=1}^{4} \lceil d_i \rceil + d_5$. By symmetry the expected value of d_5 is $\frac{4}{5}$. Also, by linearity of expectation, all we want is $4\lceil d_1 \rceil + \frac{4}{5}$. Note that probability that $\lceil d_1 \rceil \geq k$ is $(\frac{5-k}{4})^4$. So the expected value of $\lceil d_1 \rceil$ is

$$\frac{1^4 + 2^4 + 3^4 + 4^4}{4^4} = \frac{354}{4^4},$$

so the final answer is

$$\frac{354}{64} + \frac{4}{5} = \frac{177}{32} + \frac{4}{5} = \boxed{\frac{1013}{160}}.$$ □