MBMT Algebra Round — Leibniz

March 30, 2019

Full Name ____

Team Number _________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 Kev and Tim are brothers. Six years from now, Kev's age will be the square of what it is right now, and Tim's age will be the cube of what it is right now. Find the sum of Kev and Tim's ages right now.

Proposed by Haydn Gwyn

Solution. $\boxed{5}$

For Kev's age, we have $k^2 = k + 6$. For Tim's age, we have $t^3 = t + 6$. Though it is possible to solve these equations methodically, perhaps the easiest solution is to guess the solutions $k = 3$ and $t = 2$. Then we have $3 + 2 = \boxed{5}$. □

2 Shawn bought 21 apples and 9 bananas, spending a total of 45 dollars. He then proceeds to give Emmy 7 apples and 3 bananas. How many dollars does Emmy owe Shawn for the fruit?

Proposed by Daniel Zhu

Solution. $\boxed{15}$

Shawn gave Emmy $\frac{1}{3}$ of the fruit he bought, so Emmy owes $\frac{1}{3}$ of 45, which is $\boxed{15}$. □

3 Roger starts with the number 2019 on his calculator and starts hitting the square root button (which replaces the number on his calculator with its square root). How many times will he have to hit the button before the number on his calculator is less than 2.019?

Proposed by Haydn Gwyn

Solution. $\boxed{4}$

We can consider doing the problem backwards: Roger starts with the number 2.019, how many times does he have to *square* the number before the number is greater than 2019? We can prove that this new problem is equivalent, since $a < b \iff \sqrt{a} < \sqrt{b}$.

In this new problem, we approximate $2.019 \approx 2$, then start squaring:

$$2 \to 4 \to 16 \to 256 \to 65536.$$

Thus, the answer is $\boxed{4}$. □

4 Anson and Kaz are working on a group project. They must collectively complete a lab trial, write a lab journal entry, and create a Powerpoint presentation. The number of minutes each person takes to complete each task is given by the table below. They cannot work simultaneously on a task, but they can stop work partway through. How many minutes, at minimum, do Anson and Kaz need to finish all three tasks?

	Presentation	**Journal**	**Lab**
Anson	15	20	25
Kaz	30	30	15

Proposed by Kevin A. Zhou

Solution. $\boxed{27}$

Anson is the most efficient when working on the presentation, so he should be the only one working on the presentation. Similarly, Kaz should be the only one working on the lab. Both would have to work on the journal. If we let both people work for a minutes, then they will finish $\frac{a}{20} + \frac{a}{30}$ of the journal. Setting this equal to 1, we find $a = 12$. So, Anson and Kaz should work on the journal for 12 minutes each. This can be achieved using the following schedule:

Time	**Anson**	**Kaz**
12 min	Presentation	Journal
3 min	Presentation	Lab
12 min	Journal	Lab

Therefore, the two students need $\boxed{27}$ minutes to finish all 3 tasks. □

5 Let $x_1, x_2, x_3, \ldots$ be a sequence of integers such that $x_1 = 1$, $x_{n+1} = 3x_n$ for odd n, and $x_{n+1} = 2x_n$ for even n. Find the sum $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots$.

Proposed by Ambrose Yang

Solution. $\boxed{\dfrac{8}{5}}$

We note that (x_n) is a geometric sequence with common ratio $\frac{1}{6}$ for n odd and another geometric sequence for n even. Using the formula for the sum of a geometric series, we find the sum to be

$$\frac{1}{1-\frac{1}{6}} + \frac{\frac{1}{3}}{1-\frac{1}{6}} = \frac{6}{5} + \frac{2}{5} = \boxed{\frac{8}{5}}.$$ □

6 Compute

$$\left(\binom{2019}{2} - \binom{1}{2}\right)\left(\binom{2019}{2} - \binom{2}{2}\right)\left(\binom{2019}{2} - \binom{3}{2}\right)\cdots\left(\binom{2019}{2} - \binom{1000}{2}\right).$$

Note that $\binom{1}{2} = 0$.

Proposed by Daniel Zhu

Solution. $\boxed{\dfrac{3038!}{1018! \cdot 2^{1000}}}$

Note that $\binom{k}{2} = 1+2+\cdots+(k-1)$. Thus for $a > b$, $\binom{a}{2} - \binom{b}{2} = (1+\cdots+a)-(1+\cdots+b) = b+1+\ldots a = \frac{(a-b)(a+b+1)}{2}$. In this case, our expression is equal to

$$\frac{2019 \cdot 2020 \ldots 3038 \cdot 2018 \cdot 2017 \ldots 1019}{2^{1000}} = \boxed{\frac{3038!}{1018! \cdot 2^{1000}}}.$$ □

7 Compute

$$2^{2^0} + \sqrt{2^{2^1} + \sqrt{2^{2^2} + \cdots}}.$$

Proposed by Haydn Gwyn

Solution. $\boxed{3+\sqrt{5}}$

Let $x = 2^{2^0} + \sqrt{2^{2^1} + \sqrt{2^{2^2} + \cdots}}$. Observe that $\frac{x}{2} = 1 + \sqrt{1 + \sqrt{1 + \cdots}}$. Then $(\frac{x}{2} - 1)^2 = \frac{x}{2} \implies x = \boxed{3+\sqrt{5}}$ □

8 Let $f_1(x) = x/4$, $f_2(x) = x/2 + 1/4$, and $f_3(x) = x/4 + 3/4$. There exist positive integers $n_1, n_2, \ldots \in \{1, 2, 3\}$ and real numbers $x_1, x_2, \ldots$ with $|x_i| < 2019$ such that, for all positive integers k,

$$f_{n_1}(f_{n_2}(\cdots f_{n_k}(x_k)\cdots)) = \frac{1}{3}.$$

Find $n_1 + n_2 + \cdots + n_{2019}$.

Proposed by Daniel Zhu

Solution. $\boxed{4037}$

Let $g_1(x) = 4x$, $g_2(x) = 2x - 1/2$, and $g_3(x) = 4x - 3$ be the inverses of the f_i. Then we must have

$$x_n = g_{n_n}(g_{n_{n-1}}(\ldots g_{n_1}(\tfrac{1}{3})\ldots)).$$

Notice that if any x_n ever goes outside $[0, 1]$, then the x_i will escape to infinity, a contradiction. Therefore, if $n_{i+1} = 1$ then $x_i \in [0, 1/4]$, if $n_{i+1} = 2$ then $x_i \in [1/4, 3/4]$, and if $n_{i+1} = 3$ then $x_i \in [3/4, 1]$.

Now we just do computation. We get $n_1 = 2$, so $x_1 = 1/6$. Then $n_2 = 1$, so $x_2 = 2/3$. Then $n_3 = 2$, so $x_3 = 5/6$. Then $n_4 = 3$, and $x_4 = 1/3$. So the n_i repeat with a period of 4, and $n_{2019} = 8 \cdot \frac{2020}{4} - 3 = \boxed{4037}$. □