MBMT Team Round — Descartes

March 30, 2019

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **15** questions. You will have **45** minutes to complete the round. Later questions are worth more points; point values are notated next to the problem statement. (There are a total of 100 points.) Please write your answers in the simplest possible form.

DO NOT TURN THE QUESTION SHEET IN!
Use the official answer sheet.

You are highly encouraged to work with your teammates on the problems in order to solve them.

______ **1** **[4]** What is the solution to the equation $3 \cdot x \cdot 5 = 4 \cdot 5 \cdot 6$?

Proposed by Jacob Stavrianos

Solution. $\boxed{8}$

Divide both sides by 15 to get $x = 8$. ☐

______ **2** **[4]** Mr. Rose is making Platonic solids! If there are five different types of Platonic solids, and each Platonic solid can be one of three colors, how many different colored Platonic solids can Mr. Rose make?

Proposed by Daniel Monroe

Solution. $\boxed{15}$

There are 5 choices for the type of Platonic solid, and there are 3 choices for the color, so Mr. Rose can make $5 \cdot 3 = 15$ different Platonic solids. ☐

______ **3** **[4]** What fraction of the multiples of 5 between 1 and 100 inclusive are also multiples of 20?

Proposed by Steven Qu

Solution. $\boxed{\dfrac{1}{4}}$

There are 20 multiples of 5 and 5 multiples of 20 between 1 and 100 inclusive, so $\frac{5}{20} = \boxed{\frac{1}{4}}$. ☐

______ **4** **[5]** What is the maximum number of times a circle can intersect a triangle?

Proposed by Ambrose Yang

Solution. $\boxed{6}$

Each side of the triangle can intersect the circle twice. ☐

______ **5** **[5]** At an interesting supermarket, the nth apple you purchase costs n dollars, while pears are 3 dollars each. Given that Layla has exactly enough money to purchase either k apples or $2k$ pears for $k > 0$, how much money does Layla have?

Proposed by Steven Qu

Solution. $\boxed{66}$

The cost of buying k apples is $1 + \cdots + k = \frac{k(k+1)}{2}$, and the cost of buying $2k$ pears is $6k$. Setting these equal, $k(k+1) = 12k$. By inspection or algebra, $k = 11$. Therefore, Layla has $\boxed{\$66}$. ☐

6 **[5]** For how many positive integers $1 \leq n \leq 10$ does there exist a prime p such that the sum of the digits of p is n?

Proposed by Daniel Zhu

Solution. $\boxed{7}$

1 doesn't work since 10^k is not prime for $k \geq 0$. $6, 9$ don't work because of the divisibility rule for 3. Below is a table for the n that work.

n	2	3	4	5	7	8	10
p	2	3	13	5	7	17	19

Thus there are $\boxed{7}$ such n. □

7 **[6]** Real numbers a, b, c are selected uniformly and independently at random between 0 and 1. What is the probability that $a \geq b \leq c$?

Proposed by Jacob Stavrianos

Solution. $\boxed{\dfrac{1}{3}}$

Define $P(x)$ to be the probability that x is the largest number, where $x \in \{a, b, c\}$. By symmetry, $P(a) = P(b) = P(c)$. Also, there's always a largest number, so $P(a) + P(b) + P(c) = 1$. Thus, $P(b) = \boxed{\frac{1}{3}}$. □

8 **[6]** How many ordered pairs of positive integers (x, y) satisfy $\text{lcm}(x, y) = 500$?

Proposed by Daniel Zhu

Solution. $\boxed{55}$

First, factor $500 = 2^2 \cdot 5^3$. Let $x = 2^a \cdot 5^b$ and $y = 2^c \cdot 5^d$. Note that $\max(a, c) = 2, \max(b, d) = 3$. There are 5 ways to pick (a, c), and 11 ways to pick (b, d). Hence, there are $\boxed{55}$ ordered pairs in total. □

__________ **9 [7]** There are 50 dogs in the local animal shelter. Each dog is enemies with at least 2 other dogs. Steven wants to adopt as many dogs as possible, but he doesn't want to adopt any pair of enemies, since they will cause a ruckus. Considering all possible enemy networks among the dogs, find the maximum number of dogs that Steven can possibly adopt.

Proposed by Kevin A. Zhou

Solution. $\boxed{48}$

The best possible enemy network is the bipartite graph $K_{48,2}$, in other words, when there is a group of 48 dogs who are not enemies with each other, but all of them are enemies with the 2 dogs in the other group. Steven can simply adopt every dog in the group of size $\boxed{48}$. □

__________ **10 [8]** Unit circles a, b, c satisfy $d(a, b) = 1$, $d(b, c) = 2$, and $d(c, a) = 3$, where $d(x, y)$ is defined to be the minimum distance between any two points on circles x and y. Find the radius of the smallest circle entirely containing a, b, and c.

Proposed by Jacob Stavrianos

Solution. $\boxed{\dfrac{7}{2}}$

The centers of the circles form a 3-4-5 right triangle. Extend the hypotenuse to the ends of circles A, C to get a diameter of length 7, which corresponds to a radius of length $\boxed{\frac{7}{2}}$. □

__________ **11 [8]** The numbers 1 through 5 are written on a chalkboard. Every second, Sara erases two numbers a and b such that $a \geq b$ and writes $\sqrt{a^2 - b^2}$ on the board. Let M and m be the maximum and minimum possible values on the board when there is only one number left, respectively. Find the ordered pair (M, m).

Proposed by Daniel Zhu

Solution. $\boxed{(\sqrt{23}, \sqrt{3})}$

M is $\sqrt{25 - x^2}$, where x is made by applying Sara's process on $1, 2, 3, 4$ and made as small as possible. The best way to do this is $x = \sqrt{4^2 - 3^2 - 2^2 - 1^2} = \sqrt{2}$. Thus, $M = \sqrt{23}$.

To make m, we should first deal with $3, 4, 5$. If Sara does not end up with 0 after processing those numbers (by using $3^2 + 4^2 - 5^2$ or some other variant), then the remaining number on the chalkboard will be quite large. Therefore, Sara should do something like $m = \sqrt{2^2 - 1^2 - (5^2 - 4^2 - 3^2)} = \sqrt{3}$. □

12 **[9]** N people stand in a line. Bella says, "There exists an assignment of nonnegative numbers to the N people so that the sum of all the numbers is 1 and the sum of any three consecutive people's numbers does not exceed 1/2019." If Bella is right, find the minimum value of N possible.

Proposed by Steven Qu

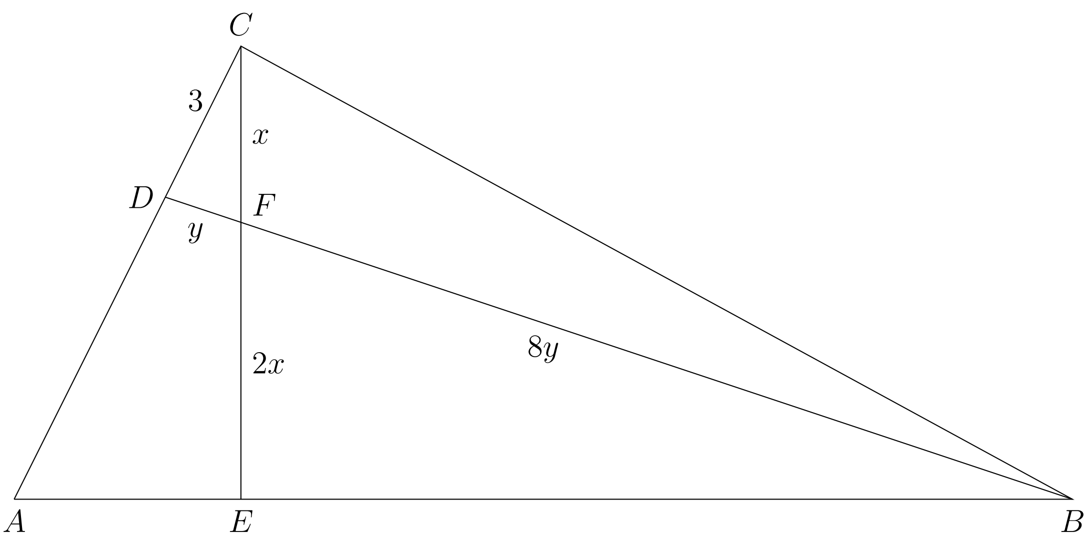
Solution. $\boxed{6055}$

Assign the numbers $\frac{1}{2019}, 0, 0, \frac{1}{2019}, 0, 0, \ldots \frac{1}{2019}$, with 2019 people assigned the number $\frac{1}{2019}$. This satisfies the problem, so $N = 6055$ is achievable. Now, suppose $N \leq 6054$. Split the N people into groups of size at most 3 by taking the first three as one group, the next three as a second group, and so on, until the last group, which may not have the full 3 people. This forms $\lceil \frac{N}{3} \rceil$ groups of size ≤ 3. By the Pigeonhole Principle, at least one of these groups will have a sum at least $\frac{1}{\lceil N/3 \rceil} \geq \frac{1}{2018} > \frac{1}{2019}$, a contradiction. Therefore, $N = 6055$ is the answer. □

13 **[9]** In triangle $\triangle ABC$, D is on AC such that BD is an altitude, and E is on AB such that CE is an altitude. Let F be the intersection of BD and CE. If $EF = 2FC$, $BF = 8DF$, and $DC = 3$, then find the area of $\triangle CDF$.

Proposed by Kevin A. Zhou

Solution. $\boxed{\dfrac{3\sqrt{3}}{2}}$



We know that $\angle CDB \cong \angle CEB$ because they are right angles. Since they both inscribe the same arc, quadrilateral $CDEB$ is cyclic. Then, by Power of a Point, $x \cdot 2x = y \cdot 8y$, or $\frac{x}{y} = 2$. This means that $\triangle CDF$ is a 30-60-90 triangle, so $DF = \sqrt{3}$. Therefore, the area is $\boxed{\dfrac{3\sqrt{3}}{2}}$. □

14 **[10]** Consider nonnegative real numbers $a_1, \ldots, a_6$ such that $a_1 + \cdots + a_6 = 20$. Find the minimum possible value of

$$\sqrt{a_1^2 + 1^2} + \sqrt{a_2^2 + 2^2} + \sqrt{a_3^2 + 3^2} + \sqrt{a_4^2 + 4^2} + \sqrt{a_5^2 + 5^2} + \sqrt{a_6^2 + 6^2}.$$

Proposed by Timothy Qian

Solution. $\boxed{29}$

Note that $\sqrt{a_i^2 + i^2}$ is the distance travelled from going up by a_i and right by i. Additionally, the entire sum is the distance of a path that goes from $(0, 0)$ to $(20, 21)$. The minimum distance of this path is $\sqrt{20^2 + 21^2} = \boxed{29}$. □

15 **[10]** Find an $a < 1000000$ so that both a and $101a$ are triangular numbers. (A triangular number is a number that can be written as $1 + 2 + \cdots + n$ for some $n \geq 1$.)

Note: There are multiple possible answers to this problem. You only need to find one.

Proposed by Daniel Zhu

Solution. $\boxed{4095 \text{ or } 6105}$

Let $101a = \frac{1}{2}n(n+1)$. One of n or $n+1$ must be divisible by 101. Since we only need to find one solution, we guess that there exists an answer with $n = 101k$, which turns out to be true. If this were not the case, we could handle the $n = 101k - 1$ case similarly.

We have $a = \frac{1}{2}k(101k + 1)$, so $m(m + 1) = k(101k + 1)$ for some m. Therefore $m^2 + m - k(101k + 1) = 0$. By the quadratic formula, for an integer solution to exist, the discriminant $1^2 + 4k(101k + 1)$ must be a perfect square. Set $r = 10$. Then we need $4r^2k^2 + 4k^2 + 4k + 1$ to be a square; call this p^2. Note that $(2rk)^2 = 4r^2k^2$, so let $q = 2rk$.

We have $p^2 - q^2 = 4k^2 + 4k + 1$. This is relatively small compared to p^2 and q^2, so we will try different values for $d = p - q$. If $d = 1$, then $2q + 1 = 4k^2 + 4k + 1 \implies 4k^2 - 36k = 0$. So $k = 9$ is a potential solution.

Under this solution $p = 2rk + 1 = 181$. So by returning to the quadratic formula, $m = \frac{1}{2}(-1 \pm 181) = 90, -91$. So $m = 90$. So $a = \frac{1}{2}m(m + 1) = \frac{1}{2}8190 = \boxed{4095}$. □