MBMT Descartes Guts Round — Set 1

March 30, 2019

__________ **1** **[3]** Find the units digit of $3^{1^{3^{3^{7}}}}$.

Proposed by Kevin A. Zhou

Solution. $\boxed{3}$

Since $1^k = 1$ for positive k, the expression is equal to $3^1 = \boxed{3}$. □

__________ **2** **[3]** Find the positive solution to the equation $x^3 - x^2 = x - 1$.

Proposed by Jacob Stavrianos

Solution. $\boxed{1}$

Factor by grouping to get $(x^2 - 1)(x - 1) = 0$. The only positive solution is 1. □

__________ **3** **[3]** Points A and B lie on a unit circle centered at O and are distance 1 apart. What is the degree measure of $\angle AOB$?

Proposed by Jacob Stavrianos

Solution. $\boxed{60}$

Triangle AOB has side lengths $1, 1, 1$, so it's equilateral. Thus, 60 degrees. □

__________ **4** **[3]** A number is a perfect square if it is equal to an integer multiplied by itself. How many perfect squares are there between 1 and 2019, inclusive?

Proposed by Daniel Monroe

Solution. $\boxed{44}$

We see that 44 is the greatest number whose squares is at most 2019, so the answer is 44. □

__________ **5** **[3]** Ted has four children of ages 10, 12, 15, and 17. In fifteen years, the sum of the ages of his children will be twice Ted's age in fifteen years. How old is Ted now?

Proposed by Olivia Fan

Solution. $\boxed{42}$

In 15 years, the sum of the ages of Ted's children will be $25 + 27 + 30 + 32 = 114$, so Tom will be age 57. Then Tom's current age is $57 - 15 = 42$. □

MBMT Descartes Guts Round — Set 2

March 30, 2019

__________ **6** **[4]** Mr. Schwartz is on the show Wipeout, and is standing on the first of 5 balls, all in a row. To reach the finish, he has to jump onto each of the balls and collect the prize on the final ball. The probability that he makes a jump from a ball to the next is 1/2, and if he doesn't make the jump he will wipe out and no longer be able to finish. Find the probability that he will finish.

Proposed by Daniel Monroe

Solution. $\boxed{1/16}$

Mr. Schwartz must make four consecutive jumps, which has a probability of $(\frac{1}{2})^4 = \frac{1}{16}$. □

__________ **7** **[4]** Kevin has written 5 MBMT questions. The shortest question is 5 words long, and every other question has exactly twice as many words as a different question. Given that no two questions have the same number of words, how many words long is the longest question?

Proposed by Jacob Stavrianos

Solution. $\boxed{80}$

The questions have length 5, 10, 20, 40, and $\boxed{80}$. □

__________ **8** **[4]** Square $ABCD$ with side length 1 is rolled into a cylinder by attaching side AD to side BC. What is the volume of that cylinder?

Proposed by Daniel Zhu

Solution. $\boxed{\dfrac{1}{4\pi}}$

The height is 1. The circumference is 1, so the radius is $\frac{1}{2\pi}$. Thus, the volume is $\boxed{\frac{1}{4\pi}}$. □

__________ **9** **[4]** Haydn is selling pies to Grace. He has 4 pumpkin pies, 3 apple pies, and 1 blueberry pie. If Grace wants 3 pies, how many different pie orders can she have?

Proposed by Steven Qu

Solution. $\boxed{7}$

There are 4 non-blueberry pie orders and 3 blueberry pie orders. □

__________ **10** **[4]** Daniel has enough dough to make 8 12-inch pizzas and 12 8-inch pizzas. However, he only wants to make 10-inch pizzas. At most how many 10-inch pizzas can he make?

Proposed by Steven Qu

Solution. $\boxed{19}$

Daniel has $(8 \cdot 12^2\pi) + (12 \cdot 8^2\pi) = 1920\pi$ square inches of dough. Each 10-inch pizza requires 100π square inches of dough. Thus, 19 pizzas. $\square$

MBMT Descartes Guts Round — Set 3

March 30, 2019

__________ **11** **[5]** A standard deck of cards contains 13 cards of each suit (clubs, diamonds, hearts, and spades). After drawing 51 cards from a randomly ordered deck, what is the probability that you have drawn an odd number of clubs?

Proposed by Jacob Stavrianos

Solution. $\boxed{\dfrac{3}{4}}$

The problem is asking for the probability that the last card remaining is not a club. Since one-fourth of the cards are clubs, that is $\boxed{\frac{3}{4}}$. □

__________ **12** **[5]** Let $s(n)$ be the sum of the digits of n. Let $g(n)$ be the number of times s must be applied to n until it has only 1 digit. Find the smallest n greater than 2019 such that $g(n) \neq g(n+1)$.

Proposed by Steven Qu

Solution. $\boxed{2025}$

Note that for all $2020 \leq k \leq 2025$, $s(k) < 10$, so $g(k) = 1$. However, $s(2026) = 10$, so $g(2026) = 2$. □

__________ **13** **[5]** In the Montgomery Blair Meterology Tournament, individuals are ranked (without ties) in ten categories. Their overall score is their average rank, and the person with the lowest overall score wins.

Alice, one of the 2019 contestants, is secretly told that her score is S. Based on this information, she deduces that she has won first place, without tying with anyone. What is the maximum possible value of S?

Proposed by Daniel Zhu

Solution. $\boxed{1.4}$

Alice's rank is the sum of her ranks divided by 10. Therefore, note that her rank is $\frac{k}{10}$, where k is an integer. If Alice has $S = 1.5$, then another contestant may also have $S_1 = 1.5$. For example, both may have 5 1st places and 5 2nd places. This is the "equality" case, so $S = 1.4$ must guarantee Alice's victory. □

__________ **14** **[5]** Let A and B be opposite vertices on a cube with side length 1, and let X be a point on that cube. Given that the distance along the surface of the cube from A to X is 1, find the maximum possible distance along the surface of the cube from B to X.

Proposed by Daniel Zhu

Solution. $\boxed{\sqrt{2}}$

We want X to "point away" from B, which is best done by having X be a vertex adjacent to A. Then it is easy to see that the distance from X to B is $\sqrt{2}$. □

15 **[5]** A function f with $f(2) > 0$ satisfies the identity $f(ab) = f(a) + f(b)$ for all $a, b > 0$. Compute $\frac{f(2^{2019})}{f(2^3)}$.

Proposed by Kevin A. Zhou

Solution. $\boxed{673}$

Substituting $a = 2$ and $b = 2^{2018}$ into the identity, we get $f(2^{2019}) = f(2) + f(2^{2018})$. Similarly, $f(2^{2019}) = f(2) + f(2) + f(2^{2017})$. It becomes clear that $f(2^{2019}) = 2019f(2)$, and that $f(8) = 3f(2)$. The quotient is $\frac{2019f(2)}{3f(2)} = \boxed{673}$.

Alternatively, observe that the function $f(x) = \log_2(x)$ satisfies the given identity, and compute $\frac{\log_2(2^{2019})}{\log_2(8)} = \boxed{673}$. □

MBMT Descartes Guts Round — Set 4

March 30, 2019

____________ **16** **[7]** Alex has 100 Bluffy Funnies in some order, which he wants to sort in order of height. They're already *almost* in order: each Bluffy Funny is at most 1 spot off from where it should be. Alex can only swap pairs of adjacent Bluffy Funnies. What is the maximum possible number of swaps necessary for Alex to sort them?

Proposed by Jacob Stavrianos

Solution. $\boxed{50}$

Let $f(n)$ be the the minimum possible number of swaps for a given almost-ordered sequence, except with n instead of 100.

Now, define Bluffy Funny i as the Funny that is i^{th} in the correct ordering. We notice that Funny 1 is in spot 1 or 2. If it's in spot 1, then $f(n) = f(n-1)$, since the 1 spot is already correct. If it's in spot 2, then Funny 2 must be in spot 1. So we swap spots 1 and 2, getting $f(n) = f(n-2) + 1$.

We're maximizing $f(n)$, so we only consider the $f(n) = f(n-2)+1$ case. $f(0) = 0$, so we get $f(2) = 1$, $f(4) = 2$, and so on until $f(2n) = n$. From this, we get $f(100) = 50$. □

____________ **17** **[7]** I start with the number 1 in my pocket. On each round, I flip a coin. If the coin lands heads heads, I double the number in my pocket. If it lands tails, I divide it by two. After five rounds, what is the expected value of the number in my pocket?

Proposed by Jacob Stavrianos

Solution. $\boxed{\dfrac{3125}{1024}}$

Solution 1. On each toss, the expected value of the number xa has a $\frac{1}{2}$ chance of becoming either $\frac{x}{2}$ or $2x$. Thus, the expected value of the number after each toss is $\frac{\frac{x}{2}+2x}{2} = \frac{5x}{4}$. Since there are 5 rounds, and the number starts as $x = 1$, the answer is $1 \cdot (\frac{5}{4})^5 = \boxed{\dfrac{3125}{1024}}$.

Solution 2 Let's do casework on how many heads I toss. Let the final number be x.

# heads	0	1	2	3	4	5
P(# heads)	1/32	5/32	5/16	5/16	5/32	1/32
x	1/32	1/8	1/2	2	8	32
$P \cdot x$	1/1024	5/256	5/32	5/8	5/4	1

The sum of the $P \cdot x$ row is $\frac{1+20+160+640+1280+1024}{1024} = \boxed{\frac{3125}{1024}}$.

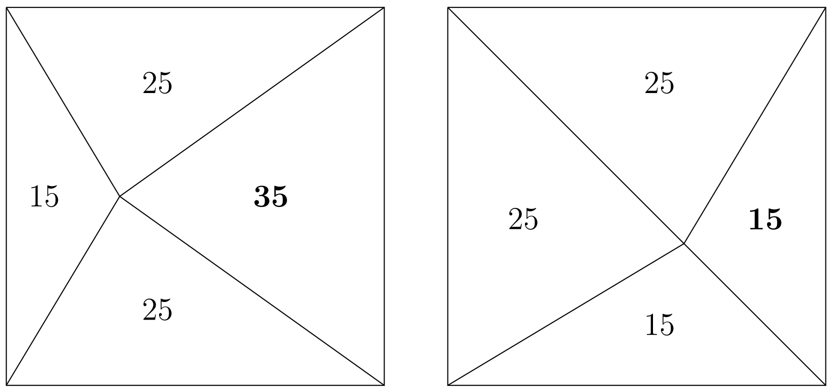
□

______ **18** **[7]** Point P inside square $ABCD$ is connected to each corner of the square, splitting the square into four triangles. If three of these triangles have area 25, 25, and 15, what are all the possible values for the area of the fourth triangle?

Proposed by Steven Qu

Solution. $\boxed{15, 35}$

Let the side length of the square be s. Then it is straightforward to show that the sum of the areas of two opposite triangles are $s^2/2$. Therefore, the sum of two of the areas must equal the sum of the other two. Casework leads to two solutions:



□

______ **19** **[7]** Mr. Stein and Mr. Schwartz are playing a yelling game. The teachers alternate yelling. Each yell is louder than the previous and is also relatively prime to the previous. If any teacher yells at 100 or more decibels, then they lose the game. Mr. Stein yells first, at 88 decibels. What volume, in decibels, should Mr. Schwartz yell at to guarantee that he will win?

Proposed by Kevin A. Zhou

Solution. $\boxed{93}$

Observe that any teacher who yells at 99 decibels will win. Therefore, any yell relatively prime to 99 decibels will lose the game, as the next teacher can yell at 99 decibels. Thus, Mr. Schwartz must yell at a volume between 88 and 99 that is relatively prime to 88 and **not** relatively prime to 99. In other words, Mr. Schwartz's yell must be a multiple of 3 but not a multiple of 2 or 11. The only number that satisfies these conditions is $\boxed{93}$. You can confirm that this number wins the game, since Mr. Stein cannot yell at a volume relatively prime to 93 but not relatively prime to 99 because this requires being a multiple of 11. □

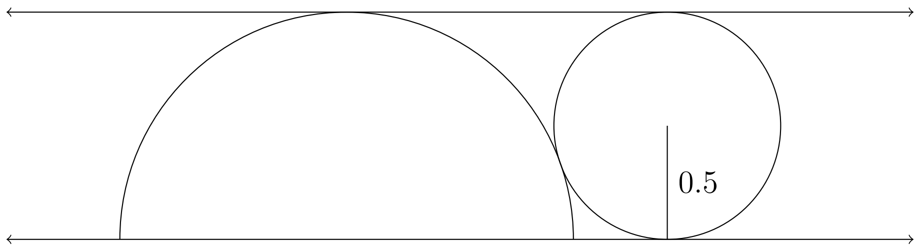
______ **20** **[7]** A semicircle of radius 1 has line ℓ along its base and is tangent to line m. Let r be the radius of the largest circle tangent to ℓ, m, and the semicircle. As the point of tangency on the semicircle varies, the range of possible values of r is the interval $[a, b]$. Find $b - a$.

Proposed by Steven Qu

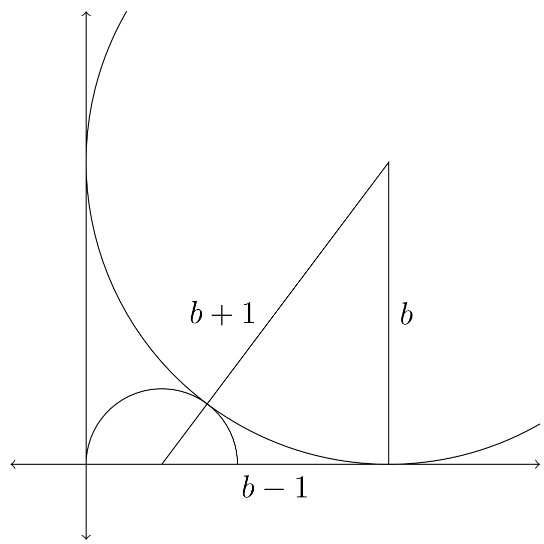
Solution. $\boxed{3.5}$

We only need the maximum and minimum values of r, which we determine to be when the point of tangency is at 0 or 90 degrees along the circle, respectively.

When m is parallel to ℓ, the radius is $a = 0.5$.



When m is perpendicular to ℓ, the radius of the circle can be found using the Pythagorean Theorem: $(b-1)^2 + b^2 = (b+1)^2$, which yields $b = 4$. Therefore, $b - a = \boxed{3.5}$.



□

MBMT Descartes Guts Round — Set 5

March 30, 2019

______________ **21** **[9]** Hungryman starts at the tile labeled "S". On each move, he moves 1 unit horizontally or vertically and eats the tile he arrives at. He cannot move to a tile he already ate, and he stops when the sum of the numbers on all eaten tiles is a multiple of nine. Find the minimum number of tiles that Hungryman eats.

S	7	9	16	18
25	27	36	45	52
54	63	70	72	81
88	90	99	108	115
117	124	126	133	135

Proposed by Kevin A. Zhou

Solution. $\boxed{18}$

Converting the table mod 9, we get:

S	7	0	7	0
7	0	0	0	7
0	0	7	0	0
7	0	0	0	7
0	7	0	7	0

Hungryman must go through all of the numbers that are $7 \pmod 9$ in order to get a sum that is $0 \pmod 9$. The shortest possible path lets Hungryman eat $\boxed{18}$ numbers. □

______________ **22** **[9]** How many triples of nonnegative integers (x, y, z) satisfy the equation $6x + 10y + 15z = 300$?

Proposed by Jacob Stavrianos

Solution. $\boxed{66}$

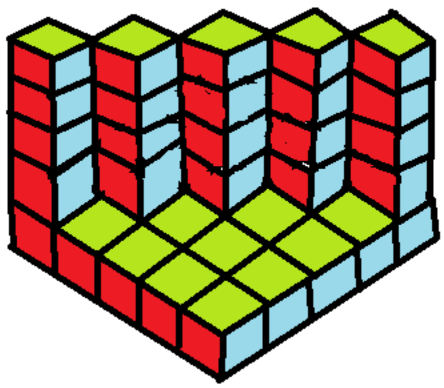
We can verify that $2 \mid z, 3 \mid y, 5 \mid x$, from which we can prove that $30 \mid 6x, 10y, 15z$. Thus, the problem is equivalent to the number of ways to distribute 10 indistinguishable "blocks" of 30 between $6x, 10y, 15x$, which by stars and bars is $\binom{12}{2} = \boxed{66}$. □

______________ **23** **[9]** Anson, Billiam, and Connor are looking at a 3D figure. The figure is made of unit cubes and is sitting on the ground. No cubes are floating; in other words, each unit cube must either have another unit cube or the ground directly under it. Anson looks from the left side and says, "I see a 5×5 square." Billiam looks from the front and says the same thing. Connor looks from the top and says the same thing. Find the absolute difference between the minimum and maximum volume of the figure.

Proposed by Kevin A. Zhou

Solution. $\boxed{80}$

A $5 \times 5 \times 5$ cube maximizes the volume, so the maximum volume is $5^3 = 125$. To minimize the volume, let the figure be a $5 \times 5 \times 1$ prism resting on the ground, with a $1 \times 1 \times 4$ pillar protruding from each of 5 unit cubes such that no two pillars share the same row or column:



Thus, the minimum volume is $5^2 + 5 \cdot 4 = 45$. The absolute difference is $\boxed{80}$. □

24 [9] Tse and Cho are playing a game. Cho chooses a number $x \in [0, 1]$ uniformly at random, and Tse guesses the value of $x(1-x)$. Tse wins if his guess is at most $\frac{1}{50}$ away from the correct value. Given that Tse plays optimally, what is the probability that Tse wins?

Proposed by Jacob Stavrianos

Solution. $\boxed{\frac{2}{5}}$

Tse should guess around where the function $x(1-x)$ is not changing much. The graph suggests that Tse should guess 0.23, so that he wins if $0.21 \le x(1-x) \le 0.25$. With some algebra (or by inspection), this occurs when $0.3 \le x \le 0.7$. Thus, the probability of Tse winning is $\boxed{\frac{2}{5}}$. □

25 [9] Find the largest solution to the equation

$$2019(x^{2019x^{2019}-2019^2+2019})^{2019} = 2019^{x^{2019}+1}.$$

Proposed by Steven Qu

Solution. $\boxed{2019^{\frac{1}{2019}}}$

Substitute $y = x^{2019}$. Then the equation becomes

$$y^{2019(y-2018)} = 2019^y.$$

Substituting $y = 2019^{a+1}$, this is

$$2019(a+1)(2019^{a+1} - 2018) = 2019^{a+1}$$

or

$$2019^a(2019a + 2018) = 2018(a+1).$$

This implies that

$$2019^{-a} = 1 + \frac{a}{2018(a+1)}.$$

Graphing both sides, it is now clear that the maximum value for a is 0. It follows that $x = \sqrt[2019]{2019}$. □

MBMT Descartes Guts Round — Set 6

March 30, 2019

This round is an estimation round. No one is expected to get an exact answer to any of these questions, but unlike other rounds, you will get points for being close. In the interest of transparency, the formulas for determining the number of points you will receive are located on the answer sheet, but they aren't very important when solving these problems.

To receive points, your answers should be positive and in decimal notation. For example, 10.55 is allowed, but not -3.2 or $\frac{2\pi}{3}$.

__________ **26** **[12]** What is the sum over all MBMT volunteers of the number of times that volunteer has attended MBMT (as a contestant or as a volunteer, including this year)?

Last year there were 47 volunteers; this is the fifth MBMT.

Proposed by Steven Qu

Solution. $\boxed{119}$

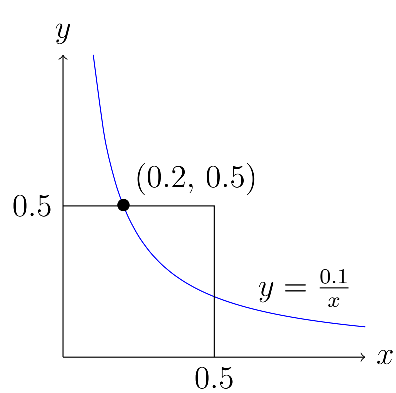
□

__________ **27** **[12]** William is sharing a chocolate bar with Naveen and Kevin. He first randomly picks a point along the bar and splits the bar at that point. He then takes the smaller piece, randomly picks a point along it, splits the piece at that point, and gives the smaller resulting piece to Kevin. Estimate the probability that Kevin gets less than 10% of the entire chocolate bar.

Proposed by Kevin A. Zhou

Solution. $\boxed{0.76651629275}$

Another way to think of this problem is: William chooses 2 random numbers $0 < x, y < 0.5$. Then, the smallest piece has size xy. We want to find the probability that $xy < 0.1$. We can think of this as throwing a dart into a square with side length 0.5, as shown below:



The area in the square that is under the blue curve is:

$$0.1 + \int_{0.2}^{0.5} \frac{0.1}{x} dx$$

Meanwhile, the total area of the square is 0.25. Some computation later, we get that the probability is $\boxed{0.7665}$. □

______ **28** **[12]** Let x be the positive solution to the equation $x^{x^{x^x}} = 1.1$. Estimate $\frac{1}{x-1}$.

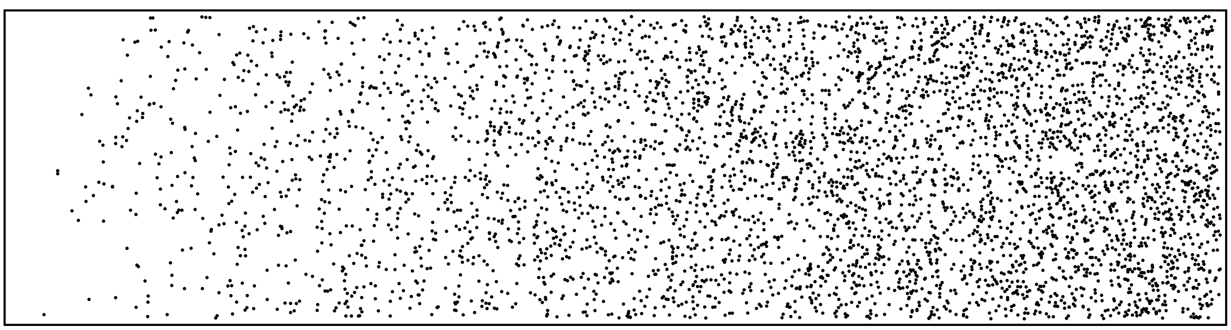
Proposed by Jacob Stavrianos

Solution. $\boxed{11.047666475590631}$

Notice that the sequence $x, x^x, x^{x^x}, \ldots$ is bounded given that x is "sufficiently small". Specifically, if we have $x^c = c$ for some positive c, then clearly $x < c$, and thus $x^x < x^c = c$, and so on. Thus, the sequence $x, x^x, x^{x^x} \ldots$ converges (quickly) to $x^{x^{x^{\cdots}}} = c \to x^c = c$.

Thus, we can instead solve $x^{1.1} = 1.1$. Applying Bernoulli's inequality, we get $x = 1.1^{\frac{1}{1.1}} \approx 1 + (.1)\frac{1}{1.1} = \frac{12}{11}$. Our approximation is then $\frac{1}{x-1} \approx 11$, which is close enough for nearly full points. □

______ **29** **[12]** Estimate the number of dots in the following box:



It may be useful to know that this image was produced by plotting $(4\sqrt{x}, y)$ some number of times, where x, y are random numbers chosen uniformly randomly and independently from the interval $[0, 1]$.

Proposed by Daniel Zhu

Solution. $\boxed{4670}$

□

____________ **30** **[12]** For a positive integer n, let $f(n)$ be the smallest prime greater than or equal to n. Estimate

$$(f(1) - 1) + (f(2) - 2) + (f(3) - 3) + \cdots + (f(10000) - 10000).$$

Proposed by Haydn Gwyn

Solution. $\boxed{57134}$

$\square$