MBMT Counting and Probability Round — Descartes

March 30, 2019

Full Name ____

Team Number _________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 Kevin Zhou rolls two six-sided dice and writes down the sum of the two numbers shown. How many numbers could Kevin have written down?

Proposed by Jacob Stavrianos

Solution. $\boxed{11}$

Kevin could have written down 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12. Thus, $\boxed{11}$. □

2 An ant is at one corner of a cube of side length 1 and can move only along the edges of the cube. How many paths of length 3 can the ant take to the opposite corner of the cube?

Proposed by Jacob Stavrianos

Solution. $\boxed{6}$

The ant has 3 choices for the first vertex it goes to. Then, the ant and its destination are on the same face of the cube. There are 2 ways for the ant to move along that face to its destination. Thus, $2 \cdot 3 = \boxed{6}$ ways. □

3 What is the probability that a given two-digit multiple of 7 has a digit sum divisible by 7?

Proposed by Haydn Gwyn

Solution. $\boxed{\dfrac{2}{13}}$

The two-digit multiples of 7 are 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98. There are 13 of these, and two of them—70 and 77—have a digit sum divisible by 7. Thus the answer is $\boxed{\frac{2}{13}}$. □

4 Felix the Frog is in the middle of an endless staircase. On every hop, he can either hop 9 steps down or 5 steps up. Felix hops 100 times. At how many possible locations can Felix end his hopping route?

Proposed by Kevin A. Zhou

Solution. $\boxed{101}$

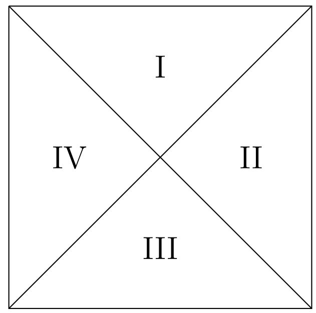
Felix can hop downward a number from 0 to 100 times. This then determines which step of the staircase he will be at. Therefore, there are $\boxed{101}$ possible locations where Felix ends his hopping route. □

5 Two points are randomly selected inside a rectangle. What is the probability that the segment connecting these two points crosses at least one of the rectangle's diagonals?

Proposed by Steven Qu

Solution. $\boxed{\dfrac{3}{4}}$

Define the following regions:



Notice that the segment crosses no diagonals if and only if the two points lie in the same region (I, II, III, or IV). By symmetry, we find that the probability of this *not* happening is $1 - \frac{1}{4} = \boxed{\frac{3}{4}}$. □

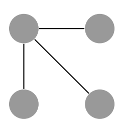
______________ **6** In a class of 4 students, everyone is friends with each other. (No one is friends with themselves, so everyone has 3 friends.) How many ways are there to break at least one of these friendships so that everyone still has an odd number of friends?

Proposed by Kevin Qian and Kevin A. Zhou

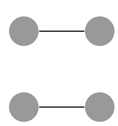
Solution. $\boxed{7}$

At least 1 friendship is broken, so someone only has 1 friend. Additionally, there cannot be more than two people with 3 friends, because then everyone would be friends with those 2+ people, and thus no one has only 1 friend. So, we have two cases.

Case 1. 1 person has 3 friends. There are 4 ways to pick the person with 3 friends. Then, all other friendships are determined:



Case 2. Nobody has 3 friends. Here, everyone has 1 friend. Therefore, there are 2 disjoint couples. There are $\binom{4}{2}$ ways to choose 1 couple, and then the other couple is determined. But, this overcounts by a factor of 2, because we could have chosen the other couple to get the same friendship network. So, there are 3 ways in this case.



In total, there are $\boxed{7}$ ways. □

7 Given a regular tetrahedron, how many ways are there to color two edges red, two edges green, and two edges blue? Rotations and reflections of a configuration are considered the same configuration.

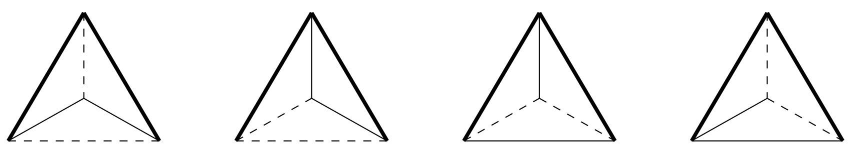
Note: A regular tetrahedron is a triangular pyramid with all faces equilateral triangles.

Proposed by Daniel Zhu

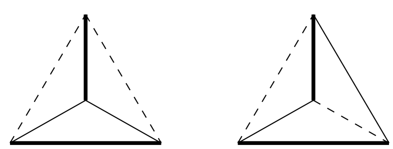
Solution. $\boxed{6}$

We will consider the tetrahedron as a 2D graph, since 3D is hard to visualize. Also, in the following diagrams, red edges are thick, green edges are normal, and blue edges are dashed.

Case 1. Red edges share a vertex. There are 4 ways.



Case 2. The red edges do not share a vertex. There are 2 ways.



□

8 Steven starts with the number 1. Then, he repeats the following procedure N times: if he has the number n, he adds a random integer from 1 to $\gcd(n, 4)$, inclusive, to n. If $N = 2019^{2019^{2019}}$, find the closest integer to $100p$, where p is the probability that Steven's final number is divisible by 4.

Proposed by Daniel Zhu

Solution. $\boxed{44}$

There are four states:

- State A: Steven's number is $0 \pmod 4$. $\frac{1}{4}$ chance of going to each of A, B, C, D
- State B: Steven's number is $1 \pmod 4$. Certain to go to C
- State C: Steven's number is $2 \pmod 4$. $\frac{1}{2}$ chance of going to each of A, D
- State D: Steven's number is $3 \pmod 4$. Go to A

Since Steven takes a near infinite number of steps ($2019^{2019^{2019}}$), the probabilities can be approximated by the steady state. We can solve for this by supposing that a, b, c, and d are the probabilities of arriving at each state in the steady state. Then

$$\begin{aligned} a &= a/4 + c/2 + d \\ b &= a/4 \\ c &= a/4 + b \\ d &= a/4 + c/2 \end{aligned}$$

So $c = a/2 \implies d = a/2$. Since $a + b + c + d = 1$, we have

$$a = \frac{1}{1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{2}} = \frac{4}{9}.$$

Thus p is extremely close to $\frac{4}{9}$, so the closet integer to $100p$ is $\boxed{44}$. $\square$