MBMT Algebra Round — Descartes

March 30, 2019

Full Name ______________________________

Team Number ______

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

______ **1** Haydn is thinking of a number. When he subtracts 20 from it and then adds 1, he gets 9. What is Haydn's number?

Proposed by Steven Qu

Solution. $\boxed{28}$

We can work backwards. Prior to adding 1, Haydn must have had 8, so his original number must have been $\boxed{28}$. □

______ **2** A daddy penguin fishes for his baby penguin, but the baby penguin is very picky about his food. On weekdays, the baby penguin demands exactly five fish per meal and eats breakfast, lunch, and dinner; on weekends, he demands five fish per meal but only eats brunch and dinner. How many fish must the daddy penguin catch in order to satisfy the baby for a week?

Proposed by Olivia Fan

Solution. $\boxed{95}$

On each of 5 weekdays, the penguin needs $3 \cdot 5 = 15$ fish. On each of 2 weekends, the penguin needs $2 \cdot 5 = 10$ fish. In total, the penguin needs $5 \cdot 15 + 2 \cdot 10 = 75 + 20 + \boxed{95}$ fish. □

______ **3** Kev and Tim are brothers. Six years from now, Kev's age will be the square of what it is right now, and Tim's age will be the cube of what it is right now. Find the sum of Kev and Tim's ages right now.

Proposed by Haydn Gwyn

Solution. $\boxed{5}$

For Kev's age, we have $k^2 = k + 6$. For Tim's age, we have $t^3 = t + 6$. Though it is possible to solve these equations methodically, perhaps the easiest solution is to guess the solutions $k = 3$ and $t = 2$. Then we have $3 + 2 = \boxed{5}$. □

______ **4** Shawn bought 21 apples and 9 bananas, spending a total of 45 dollars. He then proceeds to give Emmy 7 apples and 3 bananas. How many dollars does Emmy owe Shawn for the fruit?

Proposed by Daniel Zhu

Solution. $\boxed{15}$

Shawn gave Emmy $\frac{1}{3}$ of the fruit he bought, so Emmy owes $\frac{1}{3}$ of 45, which is $\boxed{15}$. □

______ **5** Roger starts with the number 2019 on his calculator and starts hitting the square root button (which replaces the number on his calculator with its square root). How many times will he have to hit the button before the number on his calculator is less than 2.019?

Proposed by Haydn Gwyn

Solution. $\boxed{4}$

We can consider doing the problem backwards: Roger starts with the number 2.019, how many times does he have to *square* the number before the number is greater than 2019? We can prove that this new problem is equivalent, since $a < b \iff \sqrt{a} < \sqrt{b}$.

In this new problem, we approximate $2.019 \approx 2$, then start squaring:

$$2 \to 4 \to 16 \to 256 \to 65536.$$

Thus, the answer is $\boxed{4}$. □

6 Anson and Kaz are working on a group project. They must collectively complete a lab trial, write a lab journal entry, and create a Powerpoint presentation. The number of minutes each person takes to complete each task is given by the table below. They cannot work simultaneously on a task, but they can stop work partway through. How many minutes, at minimum, do Anson and Kaz need to finish all three tasks?

	Presentation	**Journal**	**Lab**
Anson	15	20	25
Kaz	30	30	15

Proposed by Kevin A. Zhou

Solution. $\boxed{27}$

Anson is the most efficient when working on the presentation, so he should be the only one working on the presentation. Similarly, Kaz should be the only one working on the lab. Both would have to work on the journal. If we let both people work for a minutes, then they will finish $\frac{a}{20} + \frac{a}{30}$ of the journal. Setting this equal to 1, we find $a = 12$. So, Anson and Kaz should work on the journal for 12 minutes each. This can be achieved using the following schedule:

Time	**Anson**	**Kaz**
12 min	Presentation	Journal
3 min	Presentation	Lab
12 min	Journal	Lab

Therefore, the two students need $\boxed{27}$ minutes to finish all 3 tasks. □

7 Let $x_1, x_2, x_3, \dots$ be a sequence of integers such that $x_1 = 1$, $x_{n+1} = 3x_n$ for odd n, and $x_{n+1} = 2x_n$ for even n. Find the sum $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots$.

Proposed by Ambrose Yang

Solution. $\boxed{\dfrac{8}{5}}$

We note that (x_n) is a geometric sequence with common ratio $\frac{1}{6}$ for n odd and another geometric sequence for n even. Using the formula for the sum of a geometric series, we find the sum to be

$$\frac{1}{1-\frac{1}{6}} + \frac{\frac{1}{3}}{1-\frac{1}{6}} = \frac{6}{5} + \frac{2}{5} = \boxed{\frac{8}{5}}. \qquad \square$$

8 Two of the roots of $x^4 + 12x - 5$ sum to a positive rational number. Determine this sum.

Proposed by Daniel Monroe

Solution. $\boxed{2}$

If we can factor $x^4 + 12x - 5$ as $(x^2 + ax + b)(x^2 + cx + d)$, then by Vieta's formulas $-a$ or $-c$ can be the answer if they are positive.

Expanding, we need to solve the system

$$\begin{aligned} a + c &= 0 \\ b + d + ac &= 0 \\ ad + bc &= 12 \\ bd &= -5 \end{aligned}$$

Since $c = -a$, then this becomes

$$\begin{aligned} b + d &= a^2 \\ a(d - b) &= 12 \\ bd &= -5 \end{aligned}$$

If we hope that a, b, c, d are integers, then without loss of generality (b, d) is either $(-1, 5)$ or $(1, -5)$. The second case leads to $a^2 = -4$, which is impossible. The first leads to a solution $(a, b, c, d) = (2, -1, -2, 5)$. So the solution is $\boxed{2}$. $\square$