MBMT Team Round — Gauss

April 7, 2018

Full Name ______________________________________

Team Number _________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **15** questions. You will have **45** minutes to complete the round. Each question is worth the same number of points. Please write your answers in the simplest possible form.

DO NOT TURN THE QUESTION SHEET IN!
Use the official answer sheet.

You are highly encouraged to work with your teammates on the problems in order to solve them.

1 Brandon wants to maximize $\frac{\square}{\square} + \square$ by placing the numbers 1, 2, and 3 in the boxes. If each number may only be used once, what is the maximum value attainable?

Proposed by Kevin Qian

Solution. $\boxed{5}$

The denominator should be the smallest number, so we make it 1. Now, it doesn't matter where 2 or 3 goes. Either way, we get 5. $\square$

2 The planet Vriky is a sphere with radius 50 meters. Kyerk starts at the North Pole, walks straight along the surface of the sphere towards the equator, runs one full circle around the equator, and returns to the North Pole. How many meters did Kyerk travel in total throughout his journey?

Proposed by Guang Cui

Solution. $\boxed{150\pi}$

Going from the North Pole to the equator is a quarter of the great circle, so in total, Kyerk travels 1.5 times the circumference of the great circle, or 1.5*100pi = 150pi meters. $\square$

3 Mr. Pham is lazy and decides Stan's quarter grade by randomly choosing an integer from 0 to 100 inclusive. However, according to school policy, if the quarter grade is less than or equal to 50, then it is bumped up to 50. What is the probability that Stan's final quarter grade is 50?

Proposed by Kevin Qian

Solution. $\boxed{\dfrac{51}{101}}$

The final grade will be 50 if either it is already 50 or it is bumped up. The grades which are bumped up are 0 through 49, so there are 51 grades for which the final quarter grade will be 50%. Since the teacher chose from 101 possible grades, the answer is $\frac{51}{101}$ $\square$

4 We define a function f such that for all integers n, k, x, we have that

$$f(n, kx) = k^n f(n, x) \text{ and } f(n+1, x) = x f(n, x)$$

If $f(1, k) = 2k$ for all integers k, then what is $f(3, 7)$?

Proposed by Kevin Qian

Solution. $\boxed{686}$

$f(n, k) = k^n f(n, 1) = k^n f(n-1, 1) = \cdots = k^n f(1, 1) = 2k^n$. Thus, $f(3, 7) = 2 * 7^3 = 686$. $\square$

5 What is the maximum (finite) number of points of intersection between the boundaries of a equilateral triangle of side length 1 and a square of side length 20?

Proposed by Dilhan

Solution. $\boxed{4}$

First as the square is much larger than the equilateral triangle, the triangle cannot intersect opposite sides of the square. As such at most 2 sides of the square can be intersected. Now note that each side of the square can intersect the boundary of the triangle at most twice (as an equilateral triangle is convex). Therefore the maximum number of intersections is $2 \cdot 2 = \boxed{4}$. A construction is easily found (Have one side of the triangle intersect 2 sides of the square). $\square$

6 An ant is on a coordinate plane. It starts at $(0, 0)$ and takes one step each second in the North, South, East, or West direction. After 5 steps, what is the probability that the ant is at the point $(2, 1)$?

Proposed by Jyotsna Rao

Solution. $\boxed{\dfrac{25}{512}}$

To get from (0,0) to (2,1) the ant must take two steps east, and one step north. The ant must also take two other steps because it takes a total of five steps. These two other steps must cancel each other out, because otherwise the ant would end up at a different point. So, the two steps can either be north and south or east and west. If the ant goes with the first option, then it will take a total of two steps east, two steps north, and one step south. There are $\dfrac{5!}{2!\,2!\,1!} = 30$ ways to arrange these steps. If the ant goes with the second option, then it will take a total of three steps east, one step north, and one step west. There are $\dfrac{5!}{3!\,1!\,1!} = 20$ ways to arrange these steps. Thus, there are $30 + 20 = 50$ ways for the ant to get to (2,1). Since there are four choices the ant can make at each step and the ant takes five steps, there are a total of $4^5 = 1024$ ways for the ant to make the five steps. So, the probability that the ant ends up at (2,1) is $\dfrac{50}{1024} = 25/512$. $\square$

7 Find a possible solution (B, E, T) to the equation $THE + MBMT = 2018$, where T, H, E, M, B represent distinct digits from 0 to 9.

Proposed by Kevin A. Zhou

Solution. $\boxed{(8, 6, 2) \text{ OR } (7, 5, 3) \text{ OR } (4, 2, 6) \text{ OR } (3, 1, 7)}$

First, M cannot be 2 because MBMT would be greater than 2018. Therefore, M must be 1. H cannot be 9 because that requires a carry from E+T=8, which is impossible. So H = 0. From here, use casework to find values of T, B, and E that work. The solutions are T=2, B=8, E=6; T=3, B=7, E=5; T=6, B=4, E=2; T=7, B=3, E=1. $\square$

______________ **8** A sequence of positive integers is constructed such that each term is greater than the previous term, no term is a multiple of another term, and no digit is repeated in the entire sequence. An example of such a sequence would be 4, 79, 1035. How long is the longest possible sequence that satisfies these rules?

Proposed by Guang Cui

Solution. $\boxed{7}$

The sequence 4, 6, 7, 9, 10, 23, 58 gives 7.

There can't be more than 10, since there are only 10 digits. In fact, at most 9 is possible, since 0 cannot be on its own. However, 1, 2, 3, 4, 5, 6, 7, 8, 9 does not work, since 9 is a multiple of 3, etc.

104, 5, 6, 8, 9, 7, 23

1 and 0 cannot be in the sequence, and if 0 is used in a 2-digit number, 5 must be paired. Additionally, the following pairs of numbers cannot both be in the sequence: 4, 8, 3, 9, 2, 6.

From this, it is easy to deduce that there must be at least 3 two-digit numbers, or one three-digit number and a two-digit number, or a four-digit number, which makes 7 the maximum. □

______________ **9** A circle is tangent to two sides of an equilateral triangle of side length 1 as well as that same triangle's circumcircle. Find the radius of this circle.

Proposed by Steven Qu

Solution. $\boxed{\frac{2\sqrt{3}}{9}}$

This is simply the mixtilinear incircle with radius r. Let it be tangent to sides AB and AC, and let its center be M. Then let D be the foot of the altitude dropped from M to AC. MAD is a 30-60-90 triangle, MA = 2MD = 2r. However, MA is simply twice the circumradius minus r, so 2R = 3r. R = $\sqrt{3}/3$ for an equilateral triangle of side 1, so $r = 2\sqrt{3}/9$. □

______________ **10** Find the set of real numbers S so that

$$\prod_{c \in S} (x^2 + cxy + y^2) = (x^2 - y^2)(x^{12} - y^{12}).$$

Proposed by Daniel Zhu

Solution. $\boxed{\{\pm 2, \pm\sqrt{3}, \pm 1, 0\}}$

Notice

$$x^{12} - y^{12} = (x^6 - y^6)(x^6 + y^6) = (x^2 - y^2)(x^2 + y^2)(x^4 + x^2y^2 + y^2)(x^4 - x^2y^2 + y^2)$$

Now, $x^4 + x^2y^2 + y^4 = (x^2 + y^2)^2 - x^2y^2 = (x^2 + xy + y^2)(x^2 - xy + y^2)$. Similarly, $x^4 - x^2y^2 + y^4 = (x^2 + \sqrt{3}xy + y^2)(x^2 - \sqrt{3}xy + y^2)$.

Therefore, the right hand side is

$$(x^2 - y^2)^2 \prod_{c \in \{-\sqrt{3}, -1, 0, 1, \sqrt{3}\}} (x^2 + cxy + y^2).$$

Since $(x^2 - y^2)^2 = (x + y)^2(x - y)^2 = (x^2 + 2xy + y^2)(x^2 - 2xy + y^2)$ and the problem statement says that the set is unique, we are done.

Alternatively, one can consider the complex roots of both sides. □

______ **11** ABC is an equilateral triangle of side length 8. P is a point on side AB. If $AC + CP = 5 \cdot AP$, find AP.

Proposed by Steven Qu

Solution. $\boxed{3}$

Let $AP = x$. Drop a perpendicular from C to AB, call it D. $CD = 4\sqrt{3}$. We notice that $CP < 8$, so $5AP = 8 + CP < 16$, and $AP < 16/5 < 4$. In this sense, we know that P lies on segment AD. Now, consider $DP = 4 - x$. Applying the Pythagorean Theorem to right triangle CDP gives us that $(4\sqrt{3})^2 + (4 - x)^2 = (5x - 8)^2$. This is simply $x^2 + 64 - 8x = 25x^2 + 64 - 80x$, or $24x^2 = 72x$, meaning $x = 3$. □

______ **12** Given a function $f(x)$ such that $f(a + b) = f(a) + f(b) + 2ab$ and $f(3) = 0$, find $f(\frac{1}{2})$.

Proposed by Steven Qu

Solution. $\boxed{-\frac{5}{4}}$

Consider $g(x) = f(x) - x^2$. Then, substituting yields that g(a+b) = g(a) + g(b), which yields the solutions g(x) = cx over rationals for real c. Then, g(3) = -9, so c = -3, and $f(x) = -3x + x^2$, so $f(\frac{1}{2}) = -5/4$. □

______ **13** Badville is a city on the infinite Cartesian plane. It has 24 roads emanating from the origin, with an angle of 15 degrees between each road. It also has beltways, which are circles centered at the origin with any integer radius. There are no other roads in Badville. Steven wants to get from $(10, 0)$ to $(3, 3)$. What is the minimum distance he can take, only going on roads?

Proposed by Daniel Zhu

Solution. $\boxed{2 + 3\sqrt{2} + \pi}$

The closest intersections to (3, 3) are (5/sqrt(2), 5/sqrt(2)) and (2sqrt(2), 2sqrt(2)), which we will call A and B. Note that it is optimal to go on the beltway that is as close to the origin as possible, so the minimum distance from (10, 0) to A is 5 + 5pi/4 and the minimum distance from (10, 0) to B is 6 + pi. Therefore the answer is min(5 + 5pi/4 + 5 - 3sqrt(2), 6 + pi + 3sqrt(2) - 4) = min(10 + 5pi/4 - 3sqrt 2, 2 + 3sqrt 2 + pi)

The first minus the second is 8 - 6 sqrt 2 + pi / 4 6 sqrt 2 is about 1.41 * 6 = 8.46 but 8 + pi/4 is about 8 + 3/4 = 8.75

Therefore the second is smaller and the answer is 2 + pi + 3 sqrt 2. $\square$

14 Team A and Team B are playing basketball. Team A starts with the ball, and the ball alternates between the two teams. When a team has the ball, they have a 50% chance of scoring 1 point. Regardless of whether or not they score, the ball is given to the other team after they attempt to score. What is the probability that Team A will score 5 points before Team B scores any?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{2}{243}}$

Let p1 be the probability that A scores the next point, p2 that A scores the next 2 points, etc.

p1 = 1/2 + 1/4*p1 since either A scores the first time or both miss. p2 = 1/4*p1 + 1/4*p2 since either A scores and B misses (which becomes p1) or both miss. p3 = 1/4*p2 + 1/4*p3 p4 = 1/4*p3 + 1/4*p4 p5 = 1/4*p4 + 1/4*p5

Solving, p1 = 2/3, p2 = 2/9, p3 = 2/27, p4 = 2/81, and p5 = 2/243. $\square$

15 The twelve-digit integer

$$\overline{A58B3602C91D},$$

where A, B, C, D are digits with $A > 0$, is divisible by 10101. Find $\overline{ABCD}$.

Proposed by Daniel Zhu

Solution. $\boxed{4952}$

Notice that $10101 \cdot 99 = 999999$, so $10^6 \cdot 1 \pmod{10101}$.

Thus 10101 also divides

$$\overline{A58B36} + \overline{2C91D}.$$

Notice that when adding, it is impossible for there to be a carry in the ten-thousands place, so this is a six-digit integer. Since all six-digit multiples of 10101 are of the form $\overline{xyxyxy}$, this number must also be of this form.

Examining the tens places, we know that x is either 4 or 5. This means that in the thousands place, there must be a carry, so the ten-thousands place of the sum, which is y, must be 8. This forces $D = 2$, so $x = 4$. Now it is straightforward to derive that $A = 4$, $B = 9$, and $C = 5$. This answer works, as $458936 + 25912$ is indeed 484848. Thus the answer is $\boxed{4952}$. $\square$