MBMT Number Theory Round — Gauss

April 7, 2018

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

__________ **1** Haydn is playing with his toy cars. He has between 20 and 30 cars, and he knows that when he tries to put them into groups of 12, he has 1 left over. How many toy cars does Haydn have?

Proposed by Daniel Zhu

Solution. $\boxed{25}$

The number is 1 more than a multiple of 12. The only number that satisfies this and is between 20 and 30 is 25. □

__________ **2** How many 2 digit numbers $\overline{AB}$ are there so that both $\overline{AB}$ and A are divisible by 3? (If $A = 1$ and $B = 7$, then $\overline{AB} = 17$.)

Proposed by Kevin Qian

Solution. $\boxed{12}$

Since both $10a + b$ and a are divisible by 3, $10a + b - 10(a) = b$ is divisible by 3. Now, a can be 3, 6, 9 while b can be 0, 3, 6, 9, so there are $3 \times 4 = 12$ numbers. □

__________ **3** For how many ordered pairs (a, b), where a and b are positive integers, is the value 2^a8^b less than 1000?

Proposed by Jyotsna Rao

Solution. $\boxed{9}$

$2^a8^b = 2^a(2^3)^b = 2^a2^{3b} = 2^{a+3b}$. So, we want to find all (a,b) such that 2^{a+3b} is less than 1000. Since a and b are positive integers, $a + 3b$ must also be a positive integer. This means that $a + 3b \leq 9$ because $2^9 < 1000 < 2^{10}$. Since b is a positive integer, it must be at least 1. When $b = 1$, a must be between 1 and 6, so there are 6 possibilities for a. When $b = 2$, a must be between 1 and 3, so there are 3 possibilities for a. When $b \geq 3$, there are no positive solutions for a. Thus, there are $6 + 3 = 9$ possible ordered pairs. □

__________ **4** The least common multiple of two natural numbers is 8 times their greatest common factor. What is the value of the larger number divided by the smaller number?

Proposed by Jyotsna Rao

Solution. $\boxed{8}$

Let x be the value of the GCF and $8x$ the value of the LCM, and let a and b be the smaller and larger number, respectively. We know that both a and b must be divisible by x, but no other common number (other than 1). So they can be rewritten as $a = cx$ and $b = dx$, where c and d are relatively prime. The product of two numbers is equal their LCM times their GCF, so $ab = 8x^2$. This can be rewritten as $(cx)(dx) = 8x^2$, which can be simplified to $cd = 8$. Because c and d are relatively prime, d must be 8 and c must be 1. Thus, $\frac{b}{a} = \frac{dx}{cx} = \frac{8}{1}$. So, our final answer is 8. □

5 Every time DJ Khaled says "another one" he either adds a 1 to his number or appends a 1 to the end of his number. For example, he can turn 6 into 7 or 61. If he starts with the number 1, what is the minimum number of turns to get to 2018?

Proposed by Guang Cui

Solution. $\boxed{19}$

Notice that every time, a number x either becomes $x+1$ or $10x = 1$. The following sequence is possible

$$1, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 201, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018$$

To get to x0, you need (x-1)9, so this is the minimum.

Or work backwards: subtract one or take off a 1. 2018, 2017, 2016, 2015, 2014, 2013, 2012, 2011, 201, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 1 □

6 How many positive integers n less than 1000 satisfy the property that $\lfloor \sqrt[3]{n} \rfloor$ is a factor of n? Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Proposed by David Wu

Solution. $\boxed{171}$

Consider the quantity $\lfloor \sqrt[3]{n} \rfloor$. Observe that there are only 9 possible values of that quantity for $n < 1000$ (specifically the numbers 1 through 9). There are 7 possible values of n when $\lfloor \sqrt[3]{n} \rfloor = 1$, 10 values for 2, 13 for 3, 16 for 4, 19 for 5, 22 for 6, 25 for 7, 28 for 8, and 31 for 9. This gives a sum of 171 total values of n. □

7 What is the smallest positive integer n such that n divided by 7 has remainder 3, n divided by 11 has remainder 5, n divided by 13 has remainder 6, and n divided by 17 has remainder 8?

Proposed by Jason Hsu

Solution. $\boxed{8508}$

Notice $3 \cdot 2 + 1 = 7$, $5 \cdot 2 + 1 = 11$, and so on. By the Chinese Remainder Theorem, there is only one solution from 1 up to $7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Thus the answer is

$$\frac{7 \cdot 11 \cdot 13 \cdot 17 - 1}{2} = \frac{1001 \cdot 17 - 1}{2} = 8508.$$

□

8 Find

$$\gcd(2^{71} - 2, 3^{71} - 3, \ldots, 100^{71} - 100).$$

Proposed by Daniel Zhu

Solution. $\boxed{4686}$

By Fermat's Little Theorem, if $x \nmid p$, $x^{k(p-1)} \equiv 1 \pmod{p}$ for all primes p and positive integers k. Thus $p \mid x^{k(p-1)+1} - x$ for all p and k. Following this logic, all primes p such that $p - 1 \mid 70$ divide the desired gcd. Thus the gcd is a multiple of $2 \cdot 3 \cdot 11 \cdot 71 = 4686$. Now we will prove that this is the gcd.

First of all we will prove that no other primes can divide the gcd. For the sake of contradiction, assume that such a prime p exists.

If $p > 100$, then the degree-71 polynomial $x^{71} - x$ has at least 99 roots mod p. This is impossible.

If $p < 100$, then for the sake of contradiction say $p-1 \nmid 70$. Let a be the remainder when 70 is divided by $p - 1$. Then we know that $x^{71} - x = x(x^{70} - 1) \equiv x(x^a - 1) = x^{a+1} - x$. This has degree at most $p - 1$, but has p roots mod p. This is again a contradiction.

Thus the only prime factors can be the ones previously listed. To finish, notice that $p^2 \nmid p^{71} - p$, so the gcd is squarefree. $\square$