MBMT Counting and Probability Round — Gauss

April 7, 2018

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 Artemis, Zeus, and Poseidon are betting on the outcome of rolling a six-sided fair die. Artemis bets that it will be a 2, Zeus bets that it will be odd, and Poseidon bets that it will be a multiple of 3. Who is most likely to be correct?

Proposed by Daniel Zhu

Solution. $\boxed{\text{Zeus}}$

Artemis is correct if a 2 is rolled, Zeus is correct if a 1, 3, or 5 is rolled, and Poseidon is correct if a 3 or 6 is rolled. Since all rolls are equally likely, then since Zeus is correct on the most rolls, $\boxed{\text{Zeus}}$ is most likely to be correct. □

2 A pie is covered with various toppings. Strawberries cover 50% of the pie, blueberries cover 40% of the pie, and 25% of the pie has neither topping. What percentage of the pie has both toppings?

Proposed by Kevin A. Zhou

Solution. $\boxed{15\%}$

Since 25% of the pie has neither topping, 75% of the pie has at least one topping. By the principle of inclusion and exclusion, the percentage of the pie with both toppings is $50\% + 40\% - 75\% = \boxed{15\%}$. □

3 Jimmy rolls a six-sided die over and over until he gets a number less than 5. Jommy rolls a six-sided die over and over until he gets a number greater than 2. Then, they each write down their number. What is the probability that the sum of their two numbers is equal to 7?

Proposed by Haydn Gwyn

Solution. $\boxed{\frac{1}{4}}$

There are four cases for Jimmy and Jommy's numbers that sum to 7: $(1,6)$, $(2,5)$, $(3,4)$, and $(4,3)$. Since Jimmy can get 1, 2, 3, or 4 (four cases), and Jommy can get 3, 4, 5, 6 (four cases), there are $4 \cdot 4 = 16$ total cases for Jimmy and Jommy's numbers, so the probability is $\frac{4}{16} = \frac{1}{4}$. □

4 Tom is stringing red, blue, and green beads on a straight wire. A red bead can be followed by any color of bead. A blue bead can only be followed by a blue or green bead. A green bead can only be followed by a green bead. If the wire has to have 6 beads, in how many ways can Tom string the beads? Note that not all colors have to be used.

Proposed by Jyotsna Rao

Solution. $\boxed{28}$

Notice that we will always have some number of reds, followed by some number of blues, followed by some number of greens, possibly 0 of some of the colors. Therefore, we can think of this as choosing the number of beads of each color. By stars and bars, there are $\binom{8}{2} = \boxed{28}$ ways. □

____________ **5** There are 4 clubs at Blair: the Saxophone, Tambourine, Engineering, and Math clubs. Each club selects another club to be their archnemesis. What is the probability that no two clubs select each other as archnemeses?

Proposed by Kevin Qian

Solution. $\boxed{\dfrac{10}{27}}$

Count the complement where two clubs are paired. $\binom{4}{2} \cdot 3^2 - \frac{\binom{4}{2}}{2} = 51$ Thus, our answer is $1 - \frac{51}{81} = 10/27$ □

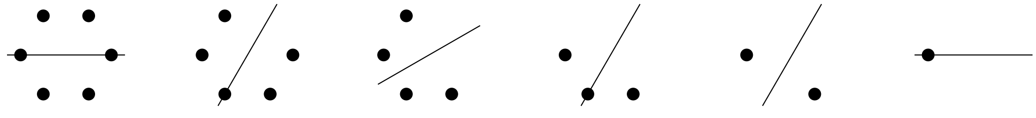
____________ **6** In a draft, several people take turns picking a person from a group to be on their team. Mr. Schwartz is picking a three-person team from a 12-person draft. Mr. Schwartz is intentionally trying to get Alice, Bob, and Claire onto his team, while the other people drafting are picking randomly. If Mr. Schwartz gets the 4th, 8th, and 10th picks, what is the probability Mr. Schwartz gets the exact team he wants?

Proposed by Dilhan Salgado

Solution. $\boxed{\dfrac{9}{88}}$

For the first three picks, the other have to pick 3 people Mr. Schwartz doesn't want, which has probability $\frac{9}{12} \cdot \frac{8}{11} \cdot \frac{7}{10}$. Mr. Schwartz then picks correctly with probability $1 = \frac{9}{9}$. Then continuing on, the probability that no-one else picks a person Mr. Schwartz wants is $\frac{6}{8} \cdot \frac{5}{7} \cdot \frac{4}{6}$, and Mr. Schwartz's probability is then $1 = \frac{5}{5}$. The other picks not going into Mr. Schwartz's group is then $3/4$. Continuing on and rearranging we can easily see that the total probability of Mr. Schwartz getting the team he wants is $\frac{9 \cdot 8 \cdot 7 \cdot 9 \cdot 6 \cdot 5 \cdot 4 \cdot 5 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4} = \frac{9 \cdot 5 \cdot 3}{12 \cdot 11 \cdot 10} = \frac{9}{88}$ □

____________ **7** Six identical chocolates are arranged in a hexagonal shape on a plate. How many ways can they be eaten, one by one, so that the shape of the remaining chocolates always has at least one line of symmetry? One order in which the chocolates could be eaten (with lines of symmetry) is shown below:



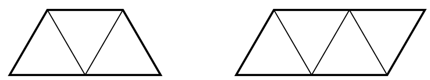
Proposed by Guang Cui

Solution. $\boxed{288 \text{ OR } 48}$

The only configuration that does not contain a line of symmetry is when there are 3 chocolates arranged in a right triangle. There are 6*2 rotations/reflections of this arrangement, 6 ways to choose from there, and 6 ways to get there, so 432 total bad orders implies there are $\boxed{288}$ good ones.

Alternatively, if you consider only the relative positions of the removals important, then we just divide by 6 to get $\boxed{48}$ □

8 Alice wants to tile an equilateral triangle of side length 10 with strips of three and four equilateral triangles (shown below) of side length 1. What is the minimum number of pieces she needs to accomplish this?



Proposed by Daniel Zhu

Solution. $\boxed{28}$

First of all, note that we need to minimize the number of strips of length 3. Decompose the big equilateral triangle into 55 equilateral triangle that point up and 45 that point down. Notice that a strip of length 4 must necessarily cover two triangles in each direction, and a strip of length 3 covers two of one orientation and one of the other. Therefore there must be at least $55 - 45 = 10$ strips of length 3. However, since there are 100 triangles in total, the number of triangles contributed by strips of length 3 must be a multiple of 4, so there must be at least 12 strips of length 3. This leads to 16 strips of length 4 and 28 pieces in total.

A construction can be found at `https://imgur.com/a/uHK64`, so 28 is optimal. □