MBMT Algebra Round — Gauss

April 7, 2018

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

______ **1** UMBC and UVA are playing a basketball game. If UMBC scores 12 three-pointers, 14 two-pointers, and 10 free throws (worth one point), while UVA scores 4 three-pointers, 19 two-pointers, and 4 free throws, then by how much did UMBC beat UVA?

Proposed by Kevin Qian

Solution. $\boxed{20}$

$$12 \times 3 + 14 \times 2 + 10 \times 1 - 4 \times 3 - 19 \times 2 - 4 \times 1 = \boxed{20}$$

□

______ **2** To test Bob's memory, Alice tells Bob the numbers 1 through 10 in some order, but she skips one number. Bob is supposed to, in return, tell Alice the skipped number. Bob doesn't have a great memory, but he is clever, so he sums up the numbers Alice tells him. If Bob gets a sum of 50, what is the missing number?

Proposed by Guang Cui

Solution. $\boxed{5}$

If no number were missing, the total sum would be $1 + ... + 10 = 55$, so the missing number is $55 - 50 = \boxed{5}$. □

______ **3** Guang's watch runs 1% slower than normal time. Luckily, he resets the time on his watch to be equal to the actual time at 6 AM, 11 AM, 4 PM, and 10 PM every day. What is the maximum difference in seconds ever achieved between the time on Guang's watch and the actual time?

Proposed by Daniel Zhu

Solution. $\boxed{288}$

Note that after a period of x hours, then Guang's watch will be $\frac{x}{100}$ hours off. Therefore the answer is $\frac{8}{100}$ hours $= \frac{480}{100}$ minutes $= \frac{24}{5}$ minutes $= 288$ seconds. □

______ **4** If $a^2 + 2b^2 = 72$ and $(a + 2b)^2 = 144$, and neither a nor b is equal to 0, find ab.

Proposed by Steven Qu

Solution. $\boxed{16}$

We have

$$(a + 2b)^2 = a^2 + 4ab + 4b^2 == 2(72) = 2a^2 + 4b^2,$$

so

$$a^2 + 4ab + 4b^2 = 2a^2 + 4b^2 \implies 4ab = a^2$$

This implies either $4b = a$ or $a = 0$. Since $a \neq 0$, $a = 4b$. Now, we plug $4b = a$ back in to get $a = 8$ and $b = 2$, so the answer is $\boxed{16}$. □

______ **5** Squares of side length x ($x < \frac{9}{2}$) are cut out of each corner of a 9 by 10 rectangular sheet of paper. The paper is then folded up into an open box. If the box has volume 60, find all possible values of x.

Proposed by Guang Cui

Solution. $\boxed{2, \dfrac{15-\sqrt{105}}{4}}$

The side lengths would be $x, 10-2x$, and $9-2x$, so $x(10-2x)(9-2x) = 60$, or $x(5-x)(9-2x) = 30$, so $2x^3 - 19x^2 + 45x - 30 = 0$. Notice that $x = 2$ is a solution, so $(x-2)(2x^2 - 15x + 15) = 0$. The quadratic has solutions $\frac{15\pm\sqrt{105}}{4}$. $\frac{15-\sqrt{105}}{4}$ is between 0 and 4.5, but $\frac{15+\sqrt{105}}{4}$ is not. Therefore, x can be $\boxed{2, \dfrac{15-\sqrt{105}}{4}}$. □

______ **6** Find the minimum value of $(x-y+1)^2 + (xy+y+1)^2$ over all pairs of real numbers (x, y).

Proposed by Kevin Qian

Solution. $\boxed{1}$

Expanding gives $(x^2+2x+2)(y^2+1) = ((x+1)^2+1)(y^2+1)$, so the answer is 1, given when $x = -1$ and $y = 0$. □

______ **7** Find the unique positive solution to $x\lceil x\lfloor x\rfloor\rceil\lceil x\rceil = 130$. Here, $\lfloor x\rfloor$ is the largest integer less than or equal to x, and $\lceil x\rceil$ is the smallest integer greater than or equal to x.

Proposed by Kevin Qian

Solution. $\boxed{\dfrac{13}{4}}$

First, observe that the function is monotonically increasing over positive x, so there is only one value of x which satisfies this. Notice that $x = 3$ is too small and $x = 4$ is too big, so our value is between 3 and 4. In particular, $\lceil x\rceil = 4$ and $\lfloor x\rfloor = 3$.

Then we can reduce the equation to

$$x\lceil 3x\rceil 4 = 130$$

Let $x = 3 + y$ where $0 < y < 1$. Then

$$(3+y)\lceil 9+3y\rceil 4 = 130$$

Now we do casework on the value of $\lceil 9+3y\rceil$.

Case 1: $\lceil 9+3y\rceil = 10$ Then we get $3 + y = \frac{130}{4\cdot 10} = \frac{13}{4}$. This gives $y = \frac{13}{4}$, which is in fact our answer.

We'll do the rest of the cases for completeness.

Case 2: $\lceil 9+3y \rceil = 11$ Then we get $3+y = \frac{130}{4\cdot 11} < 3$, so $y < 0$, a contradiction.

Case 3: $\lceil 9+3y \rceil = 12$ Then we get $3+y = \frac{130}{4\cdot 12} < 3$, so $y < 0$, a contradiction. □

____________ **8** Find the minimum value of

$$a + \frac{8}{a} + \frac{8}{a + \frac{8}{a}}$$

where a is a positive real number.

Proposed by Jyotsna Rao

Solution. $\boxed{5\sqrt{2}}$

Let $f(a) = a + \frac{8}{a}$. We see that $f(a)$ is smallest when a and $\dfrac{1}{a}$ are as close together as possible. When $a = 2\sqrt{2}$, $\dfrac{8}{a}$ is also $2\sqrt{2}$. This is the closest that a and $\dfrac{1}{a}$ can be to each other, so we know that $\min(f(a)) = 4\sqrt{2}$. By the same logic, $f(f(a))$ is smallest when $f(a)$ and $\dfrac{1}{f(a)}$ are as close together as possible. We would like to plug in $f(a) = 2\sqrt{2}$ to get the minimum value of $f(f(a))$, but the smallest value of $f(a)$ is $4\sqrt{2}$. Plugging in $f(a) = 4\sqrt{2}$ makes $f(a)$ and $\dfrac{1}{f(a)}$ as close as possible, so the minimum value of $f(f(a))$ is $f(2) = 5\sqrt{2}$. □