MBMT Team Round — Cantor

April 7, 2018

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **15** questions. You will have **45** minutes to complete the round. Each question is worth the same number of points. Please write your answers in the simplest possible form.

DO NOT TURN THE QUESTION SHEET IN!
Use the official answer sheet.

You are highly encouraged to work with your teammates on the problems in order to solve them.

______ **1** Mr. Pham flips 2018 coins. What is the difference between the maximum and minimum number of heads that can appear?

Proposed by Dilhan Salgado

Solution. $\boxed{2018}$

All or nothing. $\square$

______ **2** Brandon wants to maximize $\frac{\square}{\square} + \square$ by placing the numbers 1, 2, and 3 in the boxes. If each number may only be used once, what is the maximum value attainable?

Proposed by Kevin Qian

Solution. $\boxed{5}$

The denominator should be the smallest number, so we make it 1. Now, it doesn't matter where 2 or 3 goes. Either way, we get 5. $\square$

______ **3** Guang has 10 cents consisting of pennies, nickels, and dimes. What are all the possible numbers of pennies he could have?

Proposed by Kevin Qian

Solution. $\boxed{0, 5, 10}$

All other coins have a value that is a multiple of 5, so Guang must have a multiple of 5 number of pennies. If he have $5k$ pennies, we let the rest be nickels. This shows that Guang could have 0, 5, 10 pennies. $\square$

______ **4** The ninth edition of Campbell Biology has 1464 pages. If Chris reads from the beginning of page 426 to the end of page 449, what fraction of the book has he read?

Proposed by Chris Tong

Solution. $\boxed{\dfrac{1}{61}}$

Chris has read every page from 426 to 449, meaning that he has read $449 - 426 + 1 = 24$ pages. Therefore, the desired answer is $\frac{24}{1464} = \boxed{\dfrac{1}{61}}$ after simplifying. $\square$

______ **5** The planet Vriky is a sphere with radius 50 meters. Kyerk starts at the North Pole, walks straight along the surface of the sphere towards the equator, runs one full circle around the equator, and returns to the North Pole. How many meters did Kyerk travel in total throughout his journey?

Proposed by Guang Cui

Solution. $\boxed{150\pi}$

Going from the North Pole to the equator is a quarter of the great circle, so in total, Kyerk travels 1.5 times the circumference of the great circle, or 1.5*100pi = 150pi meters. □

______________ **6** Mr. Pham is lazy and decides Stan's quarter grade by randomly choosing an integer from 0 to 100 inclusive. However, according to school policy, if the quarter grade is less than or equal to 50, then it is bumped up to 50. What is the probability that Stan's final quarter grade is 50?

Proposed by Kevin Qian

Solution. $\boxed{\dfrac{51}{101}}$

The final grade will be 50 if either it is already 50 or it is bumped up. The grades which are bumped up are 0 through 49, so there are 51 grades for which the final quarter grade will be 50%. Since the teacher chose from 101 possible grades, the answer is $\frac{51}{101}$ □

______________ **7** What is the maximum (finite) number of points of intersection between the boundaries of a equilateral triangle of side length 1 and a square of side length 20?

Proposed by Dilhan

Solution. $\boxed{4}$

First as the square is much larger than the equilateral triangle, the triangle cannot intersect opposite sides of the square. As such at most 2 sides of the square can be intersected. Now note that each side of the square can intersect the boundary of the triangle at most twice (as an equilateral triangle is convex). Therefore the maximum number of intersections is $2 \cdot 2 = \boxed{4}$. A construction is easily found (Have one side of the triangle intersect 2 sides of the square). □

______________ **8** You enter the MBMT lottery, where contestants select three different integers from 1 to 5 (inclusive). The lottery randomly selects two winning numbers, and tickets that contain both of the winning numbers win. What is the probability that your ticket will win?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{3}{10}}$

There are 10 possible tickets (5 choose 3). To win, your ticket would contain the two winning numbers and one of the three other numbers, so 3 of 10 win. □

9 Find a possible solution (B, E, T) to the equation $THE + MBMT = 2018$, where T, H, E, M, B represent distinct digits from 0 to 9.

Proposed by Kevin A. Zhou

Solution. $\boxed{(8,6,2) \text{ OR } (7,5,3) \text{ OR } (4,2,6) \text{ OR } (3,1,7)}$

First, M cannot be 2 because MBMT would be greater than 2018. Therefore, M must be 1. H cannot be 9 because that requires a carry from E+T=8, which is impossible. So H = 0. From here, use casework to find values of T, B, and E that work. The solutions are T=2, B=8, E=6; T=3, B=7, E=5; T=6, B=4, E=2; T=7, B=3, E=1. □

10 $ABCD$ is a unit square. Let E be the midpoint of AB and F be the midpoint of AD. DE and CF meet at G. Find the area of $\triangle EFG$.

Proposed by Steven Qu

Solution. $\boxed{\dfrac{3}{40}}$

[EFG] = [EAD] – [EAF] – [DFG] = 1/4 – 1/8 – [DFG]. Note that DFG is similar to DEA with a factor of $\sqrt{5}$. Therefore, [DFG] = 1/20 and the desired answer is 1/8 – 1/20 = 3/40. □

11 The eight numbers 2015, 2016, 2017, 2018, 2019, 2020, 2021, and 2022 are split into four groups of two such that the two numbers in each pair differ by a power of 2. In how many different ways can this be done?

Proposed by Guang Cui

Solution. $\boxed{15}$

Since we only care about differences, we can shift the numbers down to 0, 1, ..., 7. We proceed by casework:

If 0 is paired with 1, then 2 can be paired with 3 (two remaining cases: 45,67 or 46,57), 4 (2 cases: 35,67 or 37,56), or 6 (2 cases).

Similarly 0 can be paired with 2 or 4 which give 5 and 4 remaining ways, respectively.

In total, there are 6+5+4 = 15 ways. □

12 We define a function f such that for all integers n, k, x, we have that

$$f(n, kx) = k^n f(n, x) \text{ and } f(n+1, x) = x f(n, x)$$

If $f(1, k) = 2k$ for all integers k, then what is $f(3, 7)$?

Proposed by Kevin Qian

Solution. $\boxed{686}$

$f(n,k) = k^n f(n,1) = k^n f(n-1,1) = \cdots = k^n f(1,1) = 2k^n$. Thus, $f(3,7) = 2 * 7^3 = 686$. □

____________ **13** A sequence of positive integers is constructed such that each term is greater than the previous term, no term is a multiple of another term, and no digit is repeated in the entire sequence. An example of such a sequence would be 4, 79, 1035. How long is the longest possible sequence that satisfies these rules?

Proposed by Guang Cui

Solution. $\boxed{7}$

The sequence 4, 6, 7, 9, 10, 23, 58 gives 7.

There can't be more than 10, since there are only 10 digits. In fact, at most 9 is possible, since 0 cannot be on its own. However, 1, 2, 3, 4, 5, 6, 7, 8, 9 does not work, since 9 is a multiple of 3, etc.

104, 5, 6, 8, 9, 7, 23

1 and 0 cannot be in the sequence, and if 0 is used in a 2-digit number, 5 must be paired. Additionally, the following pairs of numbers cannot both be in the sequence: 4, 8, 3, 9, 2, 6.

From this, it is easy to deduce that there must be at least 3 two-digit numbers, or one three-digit number and a two-digit number, or a four-digit number, which makes 7 the maximum. □

____________ **14** ABC is an equilateral triangle of side length 8. P is a point on side AB. If $AC + CP = 5 \cdot AP$, find AP.

Proposed by Steven Qu

Solution. $\boxed{3}$

Let $AP = x$. Drop a perpendicular from C to AB, call it D. $CD = 4\sqrt{3}$. We notice that $CP < 8$, so $5AP = 8 + CP < 16$, and $AP < 16/5 < 4$. In this sense, we know that P lies on segment AD. Now, consider $DP = 4 - x$. Applying the Pythagorean Theorem to right triangle CDP gives us that $(4\sqrt{3})^2 + (4-x)^2 = (5x-8)^2$. This is simply $x^2 + 64 - 8x = 25x^2 + 64 - 80x$, or $24x^2 = 72x$, meaning $x = 3$. □

____________ **15** What is the value of $(1) + (1+2) + (1+2+3) + ... + (1+2+...+49+50)$?

Proposed by Jyotsna Rao

Solution. $\boxed{22100}$

Notice that each term in this sum is a triangular number (1, 3, 6, etc.). These can all be found in the third diagonal of Pascal's triangle. To find the sum of any diagonal in Pascal's triangle, we use the hockey-stick identity. The sum of the first fifty positive integers (the last term in our sum) is the third number in the 51st row of Pascal's triangle, so our desired sum is the fourth number in the 52nd row. This can also be expressed as $\binom{52}{3}$, which is equal to 22100. $\square$