MBMT Number Theory Round — Cantor

April 7, 2018

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 Dilcue's favorite number is a one-digit positive integer. When she multiplies her favorite number by 4, the result is a two-digit number that ends in 8. What is 5 times Dilcue's favorite number?

Proposed by Guang Cui

Solution. $\boxed{35}$

Only 7 works, so $7 \cdot 5 = 35$. □

2 Anna has a perfect square number of marbles. Bob has a perfect cube number of marbles. Both have more than one marble and fewer than ten marbles. Bob has one fewer marble than Anna. How many do they have altogether?

Perfect squares are squares of integers: 0, 1, 4, etc. Perfect cubes are cubes of integers: 0, 1, 8, etc.

Proposed by Eva Quittman

Solution. $\boxed{17}$

Since both kids have between 1 and 10 marbles and they must have integer numbers (marbles are discrete objects), we look for perfect squares and cubes between 1 and 10. 9 is a perfect square and 8 is a perfect cube, and $9 - 1 = 8$, so this satisfies the requirements of the problem. Thus the answer is $9 + 8 = 17$. □

3 What is the smallest number greater than 22 that can be written as a sum of two perfect squares?

Proposed by Daniel Zhu

Solution. $\boxed{25}$

Simple casework yields that 23 and 24 cannot be written as the sum of two squares. 25 can be written as either $5^2 + 0^2$ or $3^2 + 4^2$. □

4 How many 2 digit numbers $\overline{AB}$ are there so that both $\overline{AB}$ and A are divisible by 3? (If $A = 1$ and $B = 7$, then $\overline{AB} = 17$.)

Proposed by Kevin Qian

Solution. $\boxed{12}$

Since both $10a + b$ and a are divisible by 3, $10a + b - 10(a) = b$ is divisible by 3. Now, a can be 3, 6, 9 while b can be 0, 3, 6, 9, so there are $3 \times 4 = 12$ numbers. □

5 For how many ordered pairs (a, b), where a and b are positive integers, is the value $2^a 8^b$ less than 1000?

Proposed by Jyotsna Rao

Solution. $\boxed{9}$

$2^a 8^b = 2^a (2^3)^b = 2^a 2^{3b} = 2^{a+3b}$. So, we want to find all (a,b) such that 2^{a+3b} is less than 1000. Since a and b are positive integers, $a + 3b$ must also be a positive integer. This means that $a + 3b \leq 9$ because $2^9 < 1000 < 2^{10}$. Since b is a positive integer, it must be at least 1. When $b = 1$, a must be between 1 and 6, so there are 6 possibilities for a. When $b = 2$, a must be between 1 and 3, so there are 3 possibilities for a. When $b \geq 3$, there are no positive solutions for a. Thus, there are $6 + 3 = 9$ possible ordered pairs. □

______________ **6** The least common multiple of two natural numbers is 8 times their greatest common factor. What is the value of the larger number divided by the smaller number?

Proposed by Jyotsna Rao

Solution. $\boxed{8}$

Let x be the value of the GCF and $8x$ the value of the LCM, and let a and b be the smaller and larger number, respectively. We know that both a and b must be divisible by x, but no other common number (other than 1). So they can be rewritten as $a = cx$ and $b = dx$, where c and d are relatively prime. The product of two numbers is equal their LCM times their GCF, so $ab = 8x^2$. This can be rewritten as $(cx)(dx) = 8x^2$, which can be simplified to $cd = 8$. Because c and d are relatively prime, d must be 8 and c must be 1. Thus, $\frac{b}{a} = \frac{dx}{cx} = \frac{8}{1}$. So, our final answer is 8. □

______________ **7** Guang has come up with a great time management strategy. He decides that every 91 days he will rap continuously for the entire day. Some time later, he also decides that every k days (where k is an integer) he will do math for the entire day. Assuming that Guang cannot rap and do math simultaneously, what is the minimum k for which this is possible?

Proposed by Daniel Zhu

Solution. $\boxed{7}$

If 91 and k are relatively prime, there will eventually be a day where Guang will be forced to rap and do math simultaneously, by the Chinese Remainder Theorem. Thus k must share a prime factor of 91; since $91 = 7 \cdot 13$, 7 is the smallest number which accomplishes this.

7 works if Guang raps every 13 Sundays and does math every Saturday. □

______________ **8** What is the smallest positive integer n such that n divided by 7 has remainder 3, n divided by 11 has remainder 5, and n divided by 13 has remainder 6?

Proposed by Jason Hsu

Solution. $\boxed{500}$

Notice $3 \cdot 2 + 1 = 7$, $5 \cdot 2 + 1 = 11$, and so on. By the Chinese Remainder Theorem, there is only one solution from 1 up to $7 \cdot 11 \cdot 13$. Thus the answer is

$$\frac{7 \cdot 11 \cdot 13 - 1}{2} = \frac{1001 - 1}{2} = 500.$$

$\square$