Team Number _____

MBMT Cantor Guts Round — Set 1

April 7, 2018

__________ **1** **[3]** Daniel is exactly one year younger than his friend David. If David was born in the year 2008, in what year was Daniel born?

Proposed by Dilhan Salgado

Solution. $\boxed{2009}$

One year younger means born one year later, so the answer is $\boxed{2009}$. □

__________ **2** **[3]** Mr. Pham flips three coins. What is the probability that no two coins show the same side?

Proposed by Daniel Zhu

Solution. $\boxed{0}$

This is obviously impossible, so there is a $\boxed{0}$ chance of this occurring. □

__________ **3** **[3]** John has a sheet of white paper which is 3 cm in height and 4 cm in width. He wants to paint the sky blue and the ground green so the entire paper is painted. If the ground takes up a third of the page, how much space (in cm^2) does the sky take up?

Proposed by Kevin Qian

Solution. $\boxed{8}$

The total area of the paper is 12 cm^2. Since a third is the ground and the rest is the sky, two-thirds of the paper is the sky, so the sky takes up $\frac{2}{3} \cdot 12 = \boxed{8 \text{ cm}^2}$. □

__________ **4** **[3]** Jihang and Eric are busy fidget spinning. While Jihang spins his fidget spinner at 15 revolutions per second, Eric only manages 10 revolutions per second. How many total revolutions will the two have made after 5 continuous seconds of spinning?

Proposed by Anonymous

Solution. $\boxed{125}$

They manage $15 + 10 = 25$ revolutions per second, so there are $25 \cdot 5 = \boxed{125}$ total revolutions. □

__________ **5** **[3]** Find the last digit of

$$1333337777 \cdot 209347802 \cdot 3940704 \cdot 2309476091.$$

Proposed by Daniel Zhu

Solution. $\boxed{6}$

This is the last digit of $7 \cdot 2 \cdot 4 \cdot 1 = 56$, which is $\boxed{6}$. □

Team Number _____

MBMT Cantor Guts Round — Set 2

April 7, 2018

____________ **6** **[4]** Evan, Chloe, Rachel, and Joe are splitting a cake. Evan takes $\frac{1}{3}$ of the cake, Chloe takes $\frac{1}{4}$, Rachel takes $\frac{1}{5}$, and Joe takes $\frac{1}{6}$. There is $\frac{1}{x}$ of the original cake left. What is x?

Proposed by Steven Qu

Solution. $\boxed{20}$

$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{20+15+12+10}{60}=\frac{57}{60}$. $1-\frac{57}{60}=\frac{3}{60}=\frac{1}{20}=\frac{1}{x}$, so $x=\boxed{20}$. □

____________ **7** **[4]** Pacman is a $330°$ sector of a circle of radius 4. Pacman has an eye of radius 1, located entirely inside Pacman. Find the area of Pacman, not including the eye.

Proposed by Kevin Qian

Solution. $\boxed{\dfrac{41\pi}{3}}$

The answer is

$$\frac{330}{360}\cdot\pi\cdot 4^2-\pi\cdot 1^2=\boxed{\frac{41\pi}{3}}.$$

□

____________ **8** **[4]** The sum of two prime numbers a and b is also a prime number. If $a<b$, find a.

Proposed by Kevin Qian

Solution. $\boxed{2}$

2 must also be one of the primes. Otherwise, the sum is an even number >2, which is not prime. But if 2 is one of the numbers, it must also be the smaller of the two. □

____________ **9** **[4]** A bus has 54 seats for passengers. On the first stop, 36 people get onto an empty bus. Every subsequent stop, 1 person gets off and 3 people get on. After the last stop, the bus is full. How many stops are there?

Proposed by Steven Qu

Solution. $\boxed{10}$

2 people get on at every stop - 18 people get on over 9 stops, plus the first stop. □

______ **10** **[4]** In a game, jumps are worth 1 point, punches are worth 2 points, and kicks are worth 3 points. The player must perform a sequence of 1 jump, 1 punch, and 1 kick. To compute the player's score, we multiply the 1st action's point value by 1, the 2nd action's point value by 2, the 3rd action's point value by 3, and then take the sum. For example, if we performed a punch, kick, jump, in that order, our score would be $1 \times 2 + 2 \times 3 + 3 \times 1 = 11$. What is the maximal score the player can get?

Proposed by Kevin Qian

Solution. $\boxed{14}$

One way to solve this is just try all 6 possibilities. A cooler way is to use the Rearrangement Inequality, which tells us that the maximal score is $1 \times 1 + 2 \times 2 + 3 \times 3 = 14$ $\square$

Team Number _____

MBMT Cantor Guts Round — Set 3

April 7, 2018

__________ **11** **[5]** 6 students are sitting around a circle, and each one randomly picks either the number 1 or 2. What is the probability that there will be two people sitting next to each other who pick the same number?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{31}{32}}$

The only way this doesn't happen is if the people say 121212 or 212121, which happens with probability $\frac{1}{32}$. □

__________ **12** **[5]** You can buy a single piece of chocolate for 60 cents. You can also buy a packet with two pieces of chocolate for \$1.00. Additionally, if you buy four single pieces of chocolate, the fifth one is free. What is the lowest amount of money you have to pay for 44 pieces of chocolate? Express your answer in dollars and cents (ex. \$3.70).

Proposed by Jyotsna Rao

Solution. $\boxed{\$21.20}$

A group of five single pieces of chocolate costs $4 \cdot 60 = \$2.40$, so the first 40 pieces of chocolate cost $8 \cdot 2.40 = \$19.20$. The next four pieces of chocolate cannot be bought in a group of five, so it is cheapest to buy them in two packs of two. This costs \$2.00, so the total cost is $\boxed{\$21.20}$. □

__________ **13** **[5]** For how many integers k is there an integer solution x to the linear equation $kx + 2 = 14$?

Proposed by Ambrose Yang

Solution. $\boxed{12}$

The equation $kx = 12$ must yield an integer solution x. Thus k must be a positive or negative factor of 12. 12 has 12 positive and negative factors. Therefore k can take on $\boxed{12}$ different integral values. □

__________ **14** **[5]** Ten teams face off in a swim meet. The boys teams and girls teams are ranked independently, each team receiving some number of positive integer points, and the final results are obtained by adding the points for the boys and the points for the girls. If Blair's boys got 7th place while the girls got 5th place (no ties), what is the best possible total rank for Blair?

Proposed by Guang Cui

Solution. $\boxed{\text{2nd}}$

Blair could have beaten the teams that got 8th, 9th, and 10th place for boys as well as 6th, 7th, 8th, 9th, and 10th for girls, which could be 8 different teams in the best case. (Possibility: 1st place gets 1000, 2nd 999, ..., Blair 994, 8th place 3, 9th place 2, 10th place 1, etc.) However, there must be (at least) one team that beat Blair in both boys and girls, so the best possible is $\boxed{\text{2nd}}$ place. □

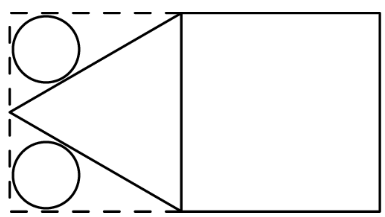
15 **[5]** Arlene has a square of side length 1, an equilateral triangle with side length 1, and two circles with radius 1/6. She wants to pack her four shapes in a rectangle without items piling on top of each other. What is the minimum possible area of the rectangle?

Proposed by Guang Cui

Solution. $\boxed{1+\dfrac{\sqrt{3}}{2}}$

The square will account for at least 1 (with no space left), while the triangle will have to use up at least $\frac{\sqrt{3}}{2}$ area (since it is inscribed in some box, and the smallest box around an equilateral triangle has area $\frac{\sqrt{3}}{2}$), so this is optimal.

The construction takes a square and a triangle on top of it, with the two circles in the spaces between the triangle and the edge of the box. To show why this works, the "empty spaces" are 30-60-90 triangles with hypotenuse 1, which has inradius $> 1/6$.



□

Team Number _____

MBMT Cantor Guts Round — Set 4

April 7, 2018

__________ **16** **[7]** Let a, b, and c be real numbers. If $a^3 + b^3 + c^3 = 64$ and $a + b = 0$, what is the value of c?

Proposed by Jyotsna Rao

Solution. $\boxed{4}$

Rearranging the second equation, we find that $b = -a$. Substituting this value for b into the first equation, we see that $a^3 - a^3 + c^3 = 64$. This means that $c^3 = 64$ and that $c = \boxed{4}$. □

__________ **17** **[7]** Bender always turns 60 degrees clockwise. He walks 3 meters, turns, walks 2 meters, turns, walks 1 meter, turns, walks 4 meters, turns, walks 1 meter, and turns. How many meters does Bender have to walk to get back to his original position?

Proposed by Guang Cui

Solution. $\boxed{2}$

One way to do it is use a grid with equilateral triangles and trace Bender's path. Another way is that there is an equiangular hexagon with side lengths 3, 2, 1, 4, 1, and x (in that order). Extending the sides and solving for x gives $\boxed{2}$. □

__________ **18** **[7]** Guang has 4 identical packs of gummies, and each pack has a red, a blue, and a green gummy. He eats all the gummies so that he finishes one pack before going on to the next pack, but he never eats two gummies of the same color in a row. How many different ways can Guang eat the gummies?

Proposed by Guang Cui

Solution. $\boxed{384}$

There are 6 ways for the first pack (RGB, RBG, etc.) For the second (and remaining) pack, only 4 of the 6 are valid, since the first color can't be the last color of the first pack. So there are $6 \cdot 4 \cdot 4 \cdot 4 = \boxed{384}$ ways. □

__________ **19** **[7]** Find the sum of all digits q such that there exists a perfect square that ends in q.

Proposed by Jesse Silverberg

Solution. $\boxed{25}$

The possible values are

$$0 + 1 + 4 + 5 + 6 + 9 = \boxed{25}.$$

□

____________ **20** **[7]** The numbers 5 and 7 are written on a whiteboard. Every minute Stev replaces the two numbers on the board with their sum and difference. After 2017 minutes the product of the numbers on the board is m. Find the number of factors of m.

Proposed by Dilhan Saigado

Solution. $\boxed{4040}$

Note that if we have a, b on the board now, in next two minutes we will transition to $a+b, a-b$ and then $2a, 2b$, so the numbers doubles every 2 minutes. After 1 minute we have $2, 12$ on the board, so after 2017 we have $2 \cdot 2^{1008}$, $12 \cdot 2^{1008}$ on the board. Therefore, the product is $3 \cdot 2^{2019}$, which has $2 \cdot 2020 = \boxed{4040}$ factors □

Team Number _____

MBMT Cantor Guts Round — Set 5

April 7, 2018

___________ **21** **[9]** On the planet Alletas, $\frac{32}{33}$ of the people with silver hair have purple eyes and $\frac{8}{11}$ of the people with purple eyes have silver hair. On Alletas, what is the ratio of the number of people with purple eyes to the number of people with silver hair?

Proposed by Kevin Qian

Solution. $\boxed{\frac{4}{3}}$

Let A = have purple eyes and B = have silver hair. Then

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{32}{33}$$
$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{8}{11}$$

Dividing the equations gives $\frac{P(A)}{P(B)} = \boxed{\frac{4}{3}}$ ☐

___________ **22** **[9]** Let P be a point on $y = -1$. Let the clockwise rotation of P by 60° about $(0, 0)$ be P'. Find the minimum possible distance between P' and $(0, -1)$.

Proposed by Kevin Qian

Solution. $\boxed{\frac{1}{2}}$

Consider the rotation of the line $y = -1$ by 60°. Then we want the shortest distance from the rotated line to $(0, -1)$. The rotated line is $y = -x\sqrt{3} - 2$. The distance from this to $(0, -1)$ is

$$\frac{-1 + 0\sqrt{3} + 2}{\sqrt{1 + 3}} = \boxed{\frac{1}{2}}$$

☐

___________ **23** **[9]** How many triangles can be made from the vertices and center of a regular hexagon? Two congruent triangles with different orientations are considered distinct.

Proposed by Jyotsna Rao

Solution. $\boxed{32}$

The hexagon has 6 vertices and 1 center, so we have 7 points from which to choose the 3 vertices of the triangle. There are $\binom{7}{3} = 35$ ways to do this. However, 3 of these "triangles" are actually just lines because they are formed from the center and opposite vertices of the hexagon. Thus, there are actually $35 - 3 = \boxed{32}$ different triangles. ☐

24 **[9]** Jeremy and Kevin are arguing about how cool a sweater is on a scale of 1–5. Jeremy says "3", and Kevin says "4". Jeremy angrily responds "3.5", to which Kevin replies "3.75". The two keep going at it, responding with the average of the previous two ratings. What rating will they converge to (and settle on as the coolness of the sweater)?

Proposed by Jeremy Zhou

Solution. $\boxed{\frac{11}{3}}$

Looking at Jeremy's ratings, we have $3, 3.5, 3.625, \ldots$. This is a converging geometric series equal to $3 + (\frac{1}{2} + \frac{1}{8} + \ldots) = 3 + \frac{\frac{1}{2}}{1-\frac{1}{4}} = \boxed{\frac{11}{3}}$. □

25 **[9]** An even positive integer n has an *odd factorization* if the largest odd divisor of n is also the smallest odd divisor of n greater than 1. Compute the number of even integers n less than 50 with an *odd factorization*.

Proposed by David Wu

Solution. $\boxed{15}$

Let $n = 2^a \cdot b$, where b is odd. Then b is the largest odd divisor of n. If b has no smaller odd divisors other than 1, then b must be prime. We now perform casework on the value of a. Clearly $a < 6$. If $a = 5$, then $b = 3$ is already too big. If $a = 4$, then $b = 3$ is the only possibility. If $a = 3$, then $b = 3$ or 5. If $a = 2$, then $b = 3$, 5, 7, or 11. If $a = 1$, then $b = 3, 5, 7, 11, 13, 17, 19$, or 23. Hence, the answer is $1 + 2 + 4 + 8 = \boxed{15}$. □

Team Number _____

MBMT Cantor Guts Round — Set 6

April 7, 2018

Note. Every answer in this section must be positive and given in decimal notation to receive points; 1000 and 4354.3 are allowed, while 2/3, -34, 0, and π are not.

Also, some of the problems are on the back of this sheet.

___________ **26** **[12]** When $2018! = 2018 \times 2017 \times \ldots \times 1$ is multiplied out and written as an integer, find the number of 4's.

If the correct answer is A and your answer is E, you will receive $12\min(A/E, E/A)^3$ points.

Proposed by Guang Cui

Solution. $\boxed{538}$

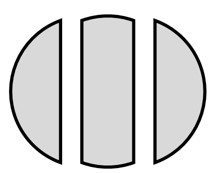
Note that the number of trailing 0's is $\lfloor \frac{2018}{5} \rfloor + \lfloor \frac{2018}{25} \rfloor + \ldots = 529$. Now, to compute the total number of digits in 2018! we can use Stirling's approximation or logs or other methods (or we may tell them how many digits there are, at least for division b). Turns out to be 5795 digits. $\frac{5795-529}{10} = 526.6$ is a good approximation, which assumes that digits are equally distributed besides the trailing zeroes.

The actual answer is $\boxed{538}$(!) and the actual distribution is

0	1	2	3	4	5	6	7	8	9
1031	503	546	564	538	562	513	508	555	475

□

___________ **27** **[12]** A circle of radius 10 is cut into three pieces of equal area with two parallel cuts. Find the width of the center piece.



If the correct answer is A and your answer is E, you will receive $\max(0, 12 - 6|A - E|)$ points.

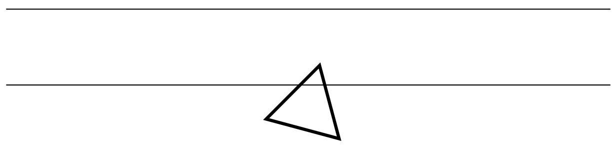
Proposed by Guang Cui

Solution. $\boxed{5.29864}$

The edge pieces should have area $\frac{100\pi}{3}$. If θ is the angle to the edge piece, then the area is equal to $100\pi \cdot \frac{\theta}{2\pi} - \frac{1}{2} \cdot 100 \cdot \sin\theta = \frac{100\pi}{3}$, so $\theta \approx 150°$.

The width is $20 \cdot \cos(\frac{\theta}{2})$, which is about $\boxed{5.299}$ □

______ **28** **[12]** An equilateral triangle of side length 1 is randomly thrown onto an infinite set of lines, spaced 1 apart.



On average, how many times will the boundary of the triangle intersect one of the lines? For example, in the above diagram, the boundary of the triangle intersects the lines in 2 places.

If the correct answer is A and your answer is E, you will receive $\max(0, 12-120|A-E|/A)$ points.

Proposed by Guang Cui

Solution. $\boxed{\dfrac{6}{\pi}}$

This is a generalization of the problem where a stick of length one is thrown (the answer is $\frac{2}{\pi}$). Suppose the stick is thrown at angle θ. Then in the band between two lines, the region where the center of the stick can be thrown and intersect some line is $\sin\theta$ of the band. Therefore, the probability is

$$\frac{\int_0^\pi \sin\theta \, d\theta}{\pi} = \frac{2}{\pi}$$

Now, by linearity of expectation, the answer is $3 \cdot \frac{2}{\pi} = \frac{6}{\pi}$.

Competitors are not expected to know this; rather they should reasonably see that it will usually intersect twice, and very rarely intersect 0 times. $\frac{6}{\pi}$ is about 1.9 □

______ **29** **[12]** Call an ordered triple of integers (a, b, c) *nice* if there exists an integer x such that $ax^2 + bx + c = 0$. How many *nice* triples are there such that $-100 \leq a, b, c \leq 100$?

If the correct answer is A and your answer is E, you will receive $12\min(A/E, E/A)$ points.

Proposed by Daniel Zhu

Solution. $\boxed{145319}$

Program it. You can use the rational root theorem to speed things up. □

____________ **30** **[12]** Let $f(i)$ denote the number of MBMT volunteers to be born in the ith state to join the United States. Find the value of $1f(1) + 2f(2) + 3f(3) + \cdots + 50f(50)$.
Note 1: Maryland was the 7th state to join the US.
Note 2: Last year's MBMT competition had 42 volunteers.

If the correct answer is A and your answer is E, you will receive $\max(0, 12 - 500(|A - E|/A)^2)$ points.

Proposed by Dilhan Salgado

Solution. $\boxed{440}$

This year there were 47 MBMT volunteers, of which 41 were born in the United States. The breakdown was

- 12 born in Maryland
- 7 born in DC (which affects the answer by 0)
- 6 born in Pennsylvania
- 3 born in California and Texas
- 2 born in New Jersey and North Carolina
- 1 born in Ohio, Michigan, Wisconsin, New York, Illinois, and Minnesota

□