Team Number _____

MBMT Ramanujan Guts Round — Set 1

April 1, 2017

__________ **1** **[3]** Find $291 + 503 - 91 + 492 - 103 - 392$.

Proposed by Guang Cui

Solution. $\boxed{700}$

$291+503-91+492-103-392 = (291-91)+(503-103)+(492-392) = 200+400+100 = \boxed{700}$. □

__________ **2** **[3]** Let the operation $a \,\&\, b$ be defined to be $\frac{a-b}{a+b}$. What is $3 \,\&\, -2$?

Proposed by Pratik Rathore

Solution. $\boxed{5}$

$3 \,\&\, -2 = \frac{3-(-2)}{3-2} = \boxed{5}$. □

__________ **3** **[3]** Joe can trade 5 apples for 3 oranges, and trade 6 oranges for 5 bananas. If he has 20 apples, what is the largest number of bananas he can trade for?

Proposed by Guang Cui

Solution. $\boxed{10}$

Joe can trade $\frac{3}{5} \cdot 20 = 12$ oranges for his 20 apples. He can then trade $\frac{5}{6} \cdot 12 = \boxed{10}$ bananas for his oranges. □

__________ **4** **[3]** A cone has a base with radius 3 and a height of 5. What is its volume? Express your answer in terms of π.

Proposed by Jyotsna Rao

Solution. $\boxed{15\pi}$

The volume is $\frac{1}{3} \cdot 3^2 \cdot 5\pi = \boxed{15\pi}$ □

__________ **5** **[3]** Guang brought dumplings to school for lunch, but by the time his lunch period comes around, he only has two dumplings left! He tries to remember what happened to the dumplings. He first traded $\frac{3}{4}$ of his dumplings for Arman's samosas, then he gave 3 dumplings to Anish, and lastly he gave David $\frac{1}{2}$ of the dumplings he had left. How many dumplings did Guang bring to school?

Proposed by David Wu

Solution. $\boxed{28}$

We work backwards. Before he gave David half of his dumplings, he must have had 4 dumplings. Then he must have had 7 dumplings before he gave three dumplings to Anish. Finally, he must have had $4 \times 7 = \boxed{28}$ dumplings before he traded with Arman. $\square$

Team Number _____

MBMT Ramanujan Guts Round — Set 2

April 1, 2017

_____ **6** **[4]** In the recording studio, Kanye has 10 different beats, 9 different manuscripts, and 8 different samples. If he must choose 1 beat, 1 manuscript, and 1 sample for his new song, how many selections can he make?

Proposed by Pratik Rathore

Solution. $\boxed{720}$

By the Fundamental Counting Principle, the answer is $10 \cdot 9 \cdot 8 = \boxed{720}$. □

_____ **7** **[4]** How many lines of symmetry does a regular dodecagon (a polygon with 12 sides) have?

Proposed by Jyotsna Rao

Solution. $\boxed{12}$

6 lines of symmetry can be drawn from a vertex to its opposite vertex and 6 lines of symmetry can be drawn as perpendicular bisectors through the dodecagon's sides, for a total of $6 + 6 = \boxed{12}$. □

_____ **8** **[4]** Let there be numbers a, b, c such that $ab = 3$ and $abc = 9$. What is the value of c?

Proposed by Pratik Rathore

Solution. $\boxed{3}$

$$c = \frac{abc}{ab} = \frac{9}{3} = \boxed{3}.$$

□

_____ **9** **[4]** How many odd composite numbers are there between 1 and 20?

Proposed by David Wu

Solution. $\boxed{2}$

There are 10 odd numbers between 1 and 20 inclusive. The primes are 3, 5, 7, 11, 13, 17, 19. Hence there are only $\boxed{2}$ odd composite numbers in this range: 9 and 15. □

_____ **10** **[4]** Consider the line given by the equation $3x - 5y = 2$. David is looking at another line of the form $ax - 15y = 5$, where a is a real number. What is the value of a such that the two lines do not intersect at any point?

Proposed by David Wu

Solution. $\boxed{9}$

The lines are parallel so that means the slopes are the same. Then $\frac{3}{5} = \frac{a}{15} \implies a = \boxed{9}$. $\square$

Team Number _____

MBMT Ramanujan Guts Round — Set 3

April 1, 2017

__________ **11** **[5]** Let $ABCD$ be a rectangle such that $AB = 4$ and $BC = 3$. What is the length of BD?

Proposed by Pratik Rathore

Solution. $\boxed{5}$

From the given lengths we also know that $CD = 4$ and $AD = 3$. The Pythagorean Theorem tells us that $BD^2 = BC^2 + CD^2$ (or $BD^2 = AD^2 + AB^2$). So $BD^2 = 3^2 + 4^2 = 25$, meaning that $BD = \boxed{5}$. □

__________ **12** **[5]** Daniel is walking at a constant rate on a 100-meter long moving walkway. The walkway moves at 3 m/s. If it takes Daniel 20 seconds to traverse the walkway, find his walking speed (excluding the speed of the walkway) in m/s.

Proposed by Guang Cui

Solution. $\boxed{2}$

Since Daniel travels 100 meters in 20 seconds, his total speed is 5 m/s. Since the walkway travels 3 m/s, Daniel travels at $5 - 3 = \boxed{2}$ m/s. □

__________ **13** **[5]** Pratik has a 6 sided die with the numbers 1, 2, 3, 4, 6, and 12 on the faces. He rolls the die twice and records the two numbers that turn up on top. What is the probability that the product of the two numbers is less than or equal to 12?

Proposed by Annie Zhao

Solution. $\boxed{\dfrac{7}{12}}$

There are 36 possibilities for the outcome of the two rolls. If he rolls a 1 on the first roll, all 6 possibilities satisfy the condition. If he rolls a 2 on the first roll, only 5 possibilities satisfy the condition. Since the numbers are factors of 12, we can easily find the number of possibilities for all other first rolls. Summing all cases, we get $\dfrac{6+5+4+3+2+1}{36} = \boxed{\dfrac{7}{12}}$. □

__________ **14** **[5]** Find the two-digit number such that the sum of its digits is twice the product of its digits.

Proposed by David Wu

Solution. $\boxed{11}$

Let the two digit number be $\overline{ab}$. Then $a + b = 2ab \implies 4ab - 2a - 2b = 0 \implies (2a-1)(2b-1) = 1 \implies a = 1, b = 1$. The number is therefore $\boxed{11}$. $\square$

______________ **15** **[5]** If $a^2 + 2a = 120$, what is the value of $2a^2 + 4a + 1$?

Proposed by Pratik Rathore

Solution. $\boxed{241}$

$2a^2 + 4a + 1 = 2(a^2 + 2a) + 1 = 2 \cdot 120 + 1 = \boxed{241}$.

Alternately, one can solve the equation for a to get $a = 10, -12$. Plugging in either one of these values into $2a^2 + 4a + 1$ gives $\boxed{241}$. $\square$

Team Number _____

MBMT Ramanujan Guts Round — Set 4

April 1, 2017

____________ **16** **[7]** Adam and Becky are building a house. Becky works twice as fast as Adam does, and they both work at constant speeds for the same amount of time each day. They plan to finish building in 6 days. However, after 2 days, their friend Charlie also helps with building the house. Because of this, they finish building in just 5 days. What fraction of the house did Adam build?

Proposed by Jyotsna Rao

Solution. $\boxed{\dfrac{5}{18}}$

Since Adam and Becky work at constant speeds and were planning to finish the house in 6 days, they together built 1/6 of the house together each day. They worked on the house for 5 days, so they built 5/6 of the house. Since Becky works twice as fast as Adam, Adam built $\frac{1}{3} \cdot \frac{5}{6} = \boxed{\dfrac{5}{18}}$ of the house. □

____________ **17** **[7]** A bag with 10 items contains both pencils and pens. Kanye randomly chooses two items from the bag, with replacement. Suppose the probability that he chooses 1 pen and 1 pencil is $\frac{21}{50}$. What are all possible values for the number of pens in the bag?

Proposed by Pratik Rathore

Solution. $\boxed{3, 7}$

Let n be the number of pens. Then there are $10 - n$ pencils. Since the items are drawn with replacement, the probability of Kane drawing 1 pen and 1 pencil is $\frac{n(10-n)}{10^2} \cdot 2 = \frac{n(10-n)}{50}$. But $\frac{n(10-n)}{50} = \frac{21}{50}$, so $n(10-n) = 21$. Expanding and rearranging gives $n^2 - 10n + 21 = 0 \implies (n-3)(n-7) = 0$. Therefore the possible values are $\boxed{3, 7}$. □

____________ **18** **[7]** In cyclic quadrilateral $ABCD$, $\angle ABD = 40°$, and $\angle DAC = 40°$. Compute the measure of $\angle ADC$ in degrees. (In cyclic quadrilaterals, opposite angles sum up to 180°.)

Proposed by David Wu

Solution. $\boxed{100}$

Since $ABCD$ is cyclic, we can inscribe it in a circle. $\angle ABD \cong \angle ACD$ since they are inscribed angles sharing the same arc. Then triangle ADC is isosceles, so $m\angle ADC = 180 - 2 \cdot 40 = \boxed{100°}$. □

19 **[7]** There is a strange random number generator which always returns a positive integer between 1 and 7500, inclusive. Half of the time, it returns a uniformly random positive integer multiple of 25, and the other half of the time, it returns a uniformly random positive integer that isn't a multiple of 25. What is the probability that a number returned from the generator is a multiple of 30?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{7}{72}}$

If the number is a multiple of 25, then there is a $1/6$ chance, since it must be divisible by 2 and 3 as well. If it isn't, then there is a $1/6 \cdot 1/6$ chance since there is a $4/24 = 1/6$ chance that the number is divisible by 5, given that it isn't divisible by 25. The answer is $1/12 + 1/72 = \boxed{\dfrac{7}{72}}$. $\square$

20 **[7]** Julia is shopping for clothes. She finds T different tops and S different skirts that she likes, where $T \geq S > 0$. Julia can either get one top and one skirt, just one top, or just one skirt. If there are 50 ways in which she can make her choice, what is $T - S$?

Proposed by Jyotsna Rao

Solution. $\boxed{14}$

Julia can get one top and one skirt in TS ways, just a top in T ways, and just a skirt in S ways. So, $TS + T + S = 50$, so by SFFT $(T+1)(S+1) = 51$. We find that one of these factors has to be 17 and the other has to be 3 for T and S to be positive integers. Since $T \geq S$, we know that $T + 1$ must be 17 and $S + 1$ must be 3. So, $T = 16, S = 2$, and $T - S = \boxed{14}$. $\square$

Team Number _____

MBMT Ramanujan Guts Round — Set 5

April 1, 2017

___________ **21** **[9]** A $5 \times 5 \times 5$ cube's surface is completely painted blue. The cube is then completely split into $1 \times 1 \times 1$ cubes. What is the average number of blue faces on each $1 \times 1 \times 1$ cube?

Proposed by Eric Lu

Solution. $\boxed{\dfrac{6}{5}}$

$5^2 \cdot 6 = 150$ faces are painted blue, and there are 125 unit cubes, so on average each cube has $\dfrac{150}{125} = \boxed{\dfrac{6}{5}}$ blue faces. □

___________ **22** **[9]** Find the number of values of n such that a regular n-gon has interior angles with integer degree measures.

Proposed by Pratik Rathore

Solution. $\boxed{22}$

The measure of an interior angle of a regular n-gon is $180(n-2)/n = 180 - 360/n$. Therefore $360/n$ must be an integer. $360 = 2^3 \cdot 3^2 \cdot 5^1$, so it has $(3+1)(2+1)(1+1) = 24$ factors. However, n cannot be 1 or 2, since all polygons have at least 3 sides. So the answer is $24 - 2 = \boxed{22}$. □

___________ **23** **[9]** 4 positive integers form an geometric sequence. The sum of the 4 numbers is 255, and the average of the second and the fourth number is 102. What is the smallest number in the sequence?

Proposed by Annie Zhao

Solution. $\boxed{3}$

Let the 4 positive integers be a, ar, ar^2, and ar^3. $\frac{a(r^4-1)}{r-1} = a(r+1)(r^2+1) = 255$. Since $204 = ar + ar^3 = ar(r^2+1)$, we have $\frac{r+1}{r} = \frac{a(r+1)(r^2+1)}{ar(r^2+1)} = \frac{255}{204} = \frac{5}{4}$. Solving gives $r = 4$. Substituting, we get $a \cdot 5 \cdot 17 = 255$, so $a = \boxed{3}$. □

___________ **24** **[9]** Let S be the set of all positive integers which have three digits when written in base 2016 and two digits when written in base 2017. Find the size of S.

Proposed by Pratik Rathore

Solution. $\boxed{4033}$

Three digits in base 2016 means n is from 2016^2 to $2016^3 - 1$. Two digits in base 2017 means n is from 2017 to $2017^2 - 1$.

Note that $2017 < 2016^2 < 2017^2 - 1 < 2016^3 - 1$.

Therefore S consists of all positive integers from 2016^2 to $2017^2 - 1$. The size of this set is $(2017^2 - 1) - (2016^2) + 1 = 2017^2 - 2016^2 = (2017 + 2016)(2017 - 2016) = \boxed{4033}$. $\square$

25 **[9]** In square $ABCD$ with side length 13, point E lies on segment CD. Segment AE divides $ABCD$ into triangle ADE and quadrilateral $ABCE$. If the ratio of the area of ADE to the area of $ABCE$ is $4 : 11$, what is the ratio of the perimeter of ADE to the perimeter of $ABCE$?

Proposed by Eric Lu

Solution. $\boxed{20 : 27}$

The triangle's area is $4/15$ of the square's. Since the square's height and the triangle's height are the same, the triangle's base is $8/15$ of the square's. Since we are dealing with ratios, let the square's side length be 15. The triangle's sides legs are 8 and 15, making the hypotenuse 17. The quadrilateral's side lengths are then 15, 15, 7, 17. The ratio is $40 : 54$ or $\boxed{20 : 27}$. $\square$

Team Number _____

MBMT Ramanujan Guts Round — Set 6

April 1, 2017

___________ **26** **[12]** Submit a decimal n to the nearest thousandth between 0 and 200. Your score will be $\min(12, S)$, where S is the non-negative difference between n and the largest number less than or equal to n chosen by another team (if you choose the smallest number, $S = n$). For example, 1.414 is an acceptable answer, while $\sqrt{2}$ and 1.4142 are not.

Proposed by Guang Cui

Solution. $\boxed{\text{N/A}}$

This is similar to Reaper. □

___________ **27** **[12]** Guang is going hard on his YNA project. From 1:00 AM Saturday to 1:00 AM Sunday, the probability that he is not finished with his project x hours after 1:00 AM on Saturday is $\frac{1}{x+1}$. If Guang does not finish by 1:00 AM on Sunday, he will stop procrastinating and finish the project immediately. Find the expected number of minutes A it will take for him to finish his project.

An estimate of E will earn $12 \cdot 2^{-|E-A|/60}$ points.

Proposed by David Wu

Solution. $\boxed{\approx 193.132549}$

The probability that Guang will not finish after x hours is $\frac{1}{x+1}$. Then the probability that he completes at time x is $dx = -\frac{1}{(x+1)^2}$. We can split it up into two cases: Either Guang finishes within the 24 hours, or he finishes it immediately at the end. The first expected value is

$$\int_0^{24} -\frac{x}{(x+1)^2} dx.$$

This evaluates to $\ln(x+1) + \frac{1}{x+1}\Big|_0^{24} = \ln(25) - \ln(1) + 1/25 - 1 = \ln(25) - 24/25$. The second expected value: It takes Guang 24 hours to complete, with probability $1/25$. Thus, the final answer is $60(\ln(25) - 24/25 + 24/25) = 60\ln(25)$. This is about 193.132549. □

___________ **28** **[12]** All the diagonals of a regular 100-gon (a regular polygon with 100 sides) are drawn. Let A be the number of distinct intersection points between all the diagonals. Find A.

An estimate of E will earn $12 \cdot (16 \log_{10}(\max(\frac{E}{A}, \frac{A}{E})) + 1)^{-\frac{1}{2}}$ or 0 points if this expression is undefined.

Proposed by David Wu

Solution. $\boxed{3731201}$

A reasonable upper bound is $\binom{100}{4}$ since every four points determine one quadrilateral and therefore one intersection of diagonals. However, there are points through which more than two diagonals pass. Thus, the answer is smaller than $\binom{100}{4}$, but not by a large amount. $\binom{100}{4} \approx 3.9 \cdot 10^6$, so an answer between $3.4 \cdot 10^6$ and $3.9 \cdot 10^6$ would be reasonable. The real answer can be found here: `http://www.wolframalpha.com/input/?i=diagonals+of+100-gon`. □

____________ **29** **[12]** Find the smallest positive integer A such that the following is true: if every integer $1, 2, \ldots, A$ is colored either red or blue, then no matter how they are colored, there are always 6 integers among them forming an increasing arithmetic progression that are all colored the same color.

An estimate of E will earn $12\min(\frac{E}{A}, \frac{A}{E})$ points or 0 points if this expression is undefined.

Proposed by Guang Cui

Solution. $\boxed{1132}$

`http://www.tandfonline.com/doi/abs/10.1080/10586458.2008.10129025` □

____________ **30** **[12]** For all integers $n \geq 2$, let $f(n)$ denote the smallest prime factor of n. Find

$$A = \sum_{n=2}^{10^6} f(n).$$

In other words, take the smallest prime factor of every integer from 2 to 10^6 and sum them all up to get A.

You may find the following values helpful: there are 78498 primes below 10^6, 9592 primes below 10^5, 1229 primes below 10^4, and 168 primes below 10^3.

An estimate of E will earn $\max(0, 12 - 4\log_{10}(\max(\frac{E}{A}, \frac{A}{E})))$ or 0 points if this expression is undefined.

Proposed by David Wu

Solution. $\boxed{37568404989}$

Getting a handle on the order of magnitude of the answer shouldn't be too bad. For example, half of the numbers are divisible by 2. For simplicity consider $f(1)$ to be in the sum even though it doesn't contribute to the sum in any way. A first order approximation yields $10^6(2 \cdot 1/2 + 3 \cdot 1/2 \cdot 1/3 + 5 \cdot 1/2 \cdot 2/3 \cdot 1/5 + \ldots)$ since for a prime p_k there are approximately

$$\prod_{i=1}^{k-1}\left(1 - \frac{1}{p_i}\right) \cdot \frac{1}{p_k}$$

numbers whose smallest prime factor is p_k.

Intuitively, these small primes should not contribute too much to the sum. Note that if n has a prime factor above 10^3, then it must be prime. Thus we first try estimate the terms up to the largest prime below 10^3.

Verifying by hand, the first few terms in the parentheses evaluate to $1 + 1/2 + 1/3 + 4/15 + 8/35 + 16/77 + ...$ Note however, that we can fudge some factors around get that the average term will be smaller than 0.2 after 8/35. So we get an upper bound of $10^6(2 + 1/5 \cdot 160) = 3.4 \cdot 10^7$.

Now we need to essentially calculate sum of primes p where $10^3 < p < 10^6$. This essentially boils down to estimating the average prime between 10^3 and 10^6. Just by looking at the provided numbers, we can tell that most of the primes are between 10^5 and 10^6. Thus, we can reasonably estimate the average prime in our given interval to be $5 \cdot 10^5$. The given information also tells us there are roughly 80000 primes between 10^3 and 10^6. Thus, these primes contribute around $8 \cdot 10^4 \times 5 \cdot 10^5 = 4 \cdot 10^{10}$ to our sum. We now see that our original terms are insignificant. Our estimate of $4 \cdot 10^{10}$ is remarkably close to the actual answer, $3.8 \cdot 10^{10}$.

$\square$