MBMT Team Round — Pascal

April 1, 2017

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **15** questions. You will have **45** minutes to complete the round. Each question is worth the same number of points. Please write your answers in the simplest possible form.

DO NOT TURN THIS TEST IN!
Use the official answer sheet.

You are highly encouraged to work with your teammates on the problems in order to solve them.

____________ **1** Let ABC be an isosceles triangle such that $\angle B = 42^\circ$. What is the sum of all possible degree measures of angle A?

Proposed by Pratik Rathore

Solution. $\boxed{207}$

$m\angle A$ can either be 42, $\frac{180-42}{2} = 69$ or $180 - 42 \cdot 2 = 96$. The sum of these values is $\boxed{207}$. □

____________ **2** Brian and Stan are playing a game. In this game, Brian rolls a fair six-sided die, while Stan rolls a fair four-sided die. Neither person shows the other what number they rolled. Brian tells Stan, "The number I rolled is guaranteed to be higher than the number you rolled." Stan now has to guess Brian's number. If Stan plays optimally, what is the probability that Stan correctly guesses the number that Brian rolled?

Proposed by Kevin Qian

Solution. $\boxed{\frac{1}{2}}$

The only way he's guaranteed to win is if the number is 5 or 6. Since each is equally likely, he has a $\boxed{\frac{1}{2}}$ chance of winning. □

____________ **3** Consider a triangle ABC with side lengths of 5, 5, and $2\sqrt{5}$. There exists a triangle with side lengths of 5, 5, and x ($x \neq 2\sqrt{5}$) which has the same area as ABC. What is the value of x?

Proposed by Pratik Rathore

Solution. $\boxed{4\sqrt{5}}$

Clearly triangle ABC is isosceles. The altitude to the side with length $2\sqrt{5}$ is $\sqrt{5^2 - \sqrt{5}^2} = 2\sqrt{5}$. If we rotate the two triangles formed by the altitude and put the $\sqrt{5}$ sides together, we get a triangle with sidelengths of 5, 5, and $4\sqrt{5}$. Therefore $x = \boxed{4\sqrt{5}}$. □

____________ **4** David is trying to write a problem for MBMT. He assigns degree measures to every interior angle in a convex n-gon, and it so happens that every angle he assigned is less than 144 degrees. He tells Pratik the value of n and the degree measures in the n-gon, and to David's dismay, Pratik claims that such an n-gon does not exist. What is the smallest value of $n \geq 3$ such that Pratik's claim is necessarily true?

Proposed by David Wu

Solution. $\boxed{10}$

If every angle in a convex n-gon is less than $180 - 360/n$, then it is impossible for it to be an n-gon. For $n = 10$, we get $180 - 360/10 = 144$, and for $n = 9$ we get $180 - 360/9 = 140 < 144$, so $n = 9$ is possible. Thus the answer is $\boxed{10}$. □

______ **5** A mother has 11 identical apples and 9 identical bananas to distribute among her 3 kids. In how many ways can the fruits be allocated so that each child gets at least one apple and one banana?

Proposed by Jyotsna Rao

Solution. $\boxed{1260}$

The mother can first distribute one apple and one banana to each child to ensure that each child gets at least one of each type of fruit. This leaves 8 apples and 6 bananas. Using stars and bars, there are $\binom{10}{2}$ ways to distribute the remaining apples and $\binom{8}{2}$ ways to distribute the remaining bananas. The answer is thus $\binom{10}{2}\binom{8}{2} = \boxed{1260}$. □

______ **6** Srinivasa Ramanujan has the polynomial $P(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$. His friend Hardy tells him that 3 is one of the roots of $P(x)$. What is the sum of the other roots of $P(x)$?

Proposed by Sambuddha Chattopadhyay

Solution. $\boxed{0}$

By Vieta's, the sum of the solutions is 3. The answer is therefore $3 - 3 = \boxed{0}$. A less experienced contestant could use the fact that one of the roots is 3, and find the other four (the remaining polynomial a product of two differences of squares, $x^2 - 1$ and $x^2 - 4$). □

______ **7** Find the sum of the five smallest positive integers that cannot be represented as the sum of two not necessarily distinct primes.

Proposed by Dilhan Salgado

Solution. $\boxed{34}$

1,2, and 3 trivially cannot be represented as the sum of two primes. The rest of the even integers up to some very large number have been verified to work. For an odd number to work it must be 2+a prime, so the ones that fail are 2+odd composite. The first two of these are $2+9 = 11$ and $2+15 = 17$. The answer is thus $1+2+3+11+17 = \boxed{34}$. □

______ **8** ABC is an equilateral triangle with side length 10. Let P be a point which lies on ray $\overrightarrow{BC}$ such that $PB = 20$. Compute the ratio $\frac{PA}{PC}$.

Proposed by Kevin Qian

Solution. $\boxed{\sqrt{3}}$

$\overline{PC} = \overline{PB} - \overline{BC} = 10$, so $\triangle PCA$ is isosceles. Since $\angle PCA = 120°$, this is a 120-30-30 triangle, so the ratio is $\boxed{\sqrt{3}}$. □

____________ **9** Let ABC be a triangle such that $AB = 10$, $BC = 14$, and $AC = 6$. The median CD and angle bisector CE are both drawn to side AB. What is the ratio of the area of triangle CDE to the area of triangle ABC?

Proposed by Pratik Rathore

Solution. $\boxed{\frac{1}{5}}$

$AD = 5$ since CD is a median. From the angle bisector theorem, $AE = 3$. So $DE = |AE - AD| = 2$. Since $\triangle CDE$ and $\triangle ABC$ share the same altitude,

$$\frac{[CDE]}{[ABC]} = \frac{2}{10} = \boxed{\frac{1}{5}}.$$

□

____________ **10** Find all integer values of x between 0 and 2017 inclusive, which satisfy

$$2016x^{2017} + 990x^{2016} + 2x + 17 \equiv 0 \pmod{2017}.$$

Proposed by Annie Zhao

Solution. $\boxed{1010}$

It is clear that $x = 0$ does not work. Then by Fermat's Little Theorem, $x^{2016} \equiv 1 \pmod{2017}$, since 2017 is prime. Therefore, the problem can be reduced to $2018x + 1007 \equiv 0 \pmod{2017}$, so $x \equiv \boxed{1010} \pmod{2017}$. □

____________ **11** Let $x^2 + ax + b$ be a quadratic polynomial with positive integer roots such that $a^2 - 2b = 97$. Compute $a + b$.

Proposed by Pratik Rathore

Solution. $\boxed{23}$

Let the roots of the polynomial be r_1 and r_2. From Vieta's we can see that $a^2 - 2b = (r_1 + r_2)^2 - 2r_1r_2 = r_1^2 + r_2^2$. Testing the cases, we find that the only possible values for (r_1, r_2) are $(9, 4)$ and $(4, 9)$. Either way, $a = -13$ and $b = 36$, meaning that the answer is $-13 + 36 = \boxed{23}$. □

12 Let S be the set $\{2, 3, \ldots, 14\}$. We assign a distinct number from S to each side of a six-sided die. We say a numbering is predictable if prime numbers are always opposite prime numbers and composite numbers are always opposite composite numbers. How many predictable numberings are there? (Rotations of a die are not distinct)

Proposed by David Wu

Solution. $\boxed{5280}$

There are 6 prime and 7 composite numbers in S. We disregard rotations until the end, dividing by 24 (since each face can be at the top and there are 4 ways to rotate it from there). If they are all prime pairs, there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ ways. If they are all composite, there are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 7!$ ways. If there are 2 prime pairs and 1 composite pair, there are $3(6 \cdot 5 \cdot 4 \cdot 3 \cdot 7 \cdot 6)$ ways. If there are 2 composite pairs and 1 prime pair, there are $3(7 \cdot 6 \cdot 5 \cdot 4 \cdot 6 \cdot 5)$ ways.

The final answer is

$$\frac{6! + 7! + 3(7 \cdot 6 \cdot 6 \cdot 5 \cdot 5 \cdot 4 + 7 \cdot 6 \cdot 6 \cdot 5 \cdot 4 \cdot 3)}{24} =$$

$$30 + 210 + 3(7 \cdot 6 \cdot 5 \cdot 5 + 7 \cdot 6 \cdot 5 \cdot 3) =$$

$$\boxed{5280}$$

□

13 In triangle ABC, $AB = 10$, $BC = 21$, and $AC = 17$. D is the foot of the altitude from A to BC, E is the foot of the altitude from D to AB, and F is the foot of the altitude from D to AC. Find the area of the smallest circle that contains the quadrilateral $AEDF$.

Proposed by Guang Cui

Solution. $\boxed{16\pi}$

Notice that $AD = 8$ (this can be seen by observing right triangles or using Heron's formula). Since $AEDF$ is a cyclic quadrilateral, and AD is the diameter, the radius of the circumcircle is 4, so the area is $\boxed{16\pi}$. □

14 What is the greatest distance between any two points on the graph of

$$3x^2 + 4y^2 + z^2 - 12x + 8y + 6z = -11?$$

Proposed by Jyotsna Rao

Solution. $\boxed{2\sqrt{14}}$

The equation can be rewritten as $3/14(x-2)^2 + 2/7(y+1)^2 + 1/14(z+3)^2 = 1$. This makes it more clear that the figure is an ellipsoid. We know that the largest distance between two points on the ellipsoid must be the length of one of its principal axes. The ellipsoid's principal axes have lengths $2\sqrt{14/3}$, $2\sqrt{7/2}$, and $2\sqrt{14}$. The largest of these is $\boxed{2\sqrt{14}}$. □

15 For a positive integer n, $\tau(n)$ is defined to be the number of positive divisors of n. Given this information, find the largest positive integer n less than 1000 such that

$$\sum_{d|n} \tau(d) = 108$$

In other words, we take the sum of $\tau(d)$ for every positive divisor d of n, which has to be 108.

Proposed by Pratik Rathore

Solution. $\boxed{980}$

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$. Then the sum in question can be written as

$$(1+2+...+(e_1+1))(1+2+...+(e_2+2))\cdots(1+2+...+(e_m+1)).$$

Therefore, we want to find sets of triangular numbers that multiply to 108.

The triangular numbers up to 108 are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105. We can quickly observe that only $3 \cdot 6 \cdot 6$ and $3 \cdot 36$ give 108. Thus n is of the form pq^2r^2 or pq^7, where p, q, r are distinct primes.

Case 1: $n = pq^2r^2$

- $p = 2 \implies$ Max is $2 \cdot 3^2 \cdot 7^2 = 882$
- $p = 3 \implies$ Max is $3 \cdot 7^2 \cdot 2^2 = 588$
- $p = 5 \implies$ Max is $5 \cdot 7^2 \cdot 2^2 = 980$
- $p = 7 \implies$ Max ss $7 \cdot 5^2 \cdot 2^2 = 700$

If $p > 7$, then the max for q^2r^2 becomes $3^2 \cdot 2^2 = 36$. Since n is less than 1000, the maximum value for p is then 23. $23 \cdot 3^2 \cdot 2^2 = 828$, which is less than 980.

Case 2: pq^7
$q = 2$ due to our bounds on n. This means that the max for this case is $7 \cdot 2^7 = 896$.

Looking at the results of both cases, the answer must be $\boxed{980}$. □