MBMT Number Theory Round — Pascal

April 1, 2017

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered by fewer competitors will be weighted more heavily. Please write your answers in the simplest possible form.

______________ **1** What is the greatest common factor of 91 and 78?

Proposed by David Wu

Solution. $\boxed{13}$

Without using the Euclidean Algorithm, we just factor 91 and 78. $91 = 7 \cdot 13$, and $78 = 6 \cdot 13$. Hence the answer is $\boxed{13}$. □

______________ **2** Let $\overline{201A}$ be a four-digit number that is divisible by 3. Find the sum of all possible values of A.

Proposed by Pratik Rathore

Solution. $\boxed{18}$

Since the sum of the digits of the number is $3 + A$, A can be any multiple of 3, so the answer is $0 + 3 + 6 + 9 = \boxed{18}$. □

______________ **3** How many three-digit positive integers are divisible either by 3 or by 7, but not by both?

Proposed by Jyotsna Rao

Solution. $\boxed{342}$

The number of three-digit positive integers divisible by 3 is $\lfloor 999/3 \rfloor - \lfloor 100/3 \rfloor = 300$. The number of three-digit positive integers divisible by both 3 and 7 is $\lfloor 999/21 \rfloor - \lfloor 100/21 \rfloor = 43$. So, the number of three-digit positive integers divisible by 3 but not 7 is $300 - 43 = 257$. The number of three-digit positive integers divisible by 7 is $\lfloor 999/7 \rfloor - \lfloor 100/7 \rfloor = 128$. So, the number of three-digit positive integers divisible by 7 but not 3 is $128 - 43 = 85$. The final answer is thus $257 + 85 = \boxed{342}$. □

______________ **4** Benedict Arnold is a confused man. If he eats a George Washington cake he loses a traitor point. But if he eats a George Wilhelm cake he doubles his traitor points. If he reaches exactly 2017 traitor points, Ethan Allen won't buy him furniture. If Benedict Arnold starts out with 1 traitor point, what is the minimum number of cakes he must eat so that Ethan Allen won't buy him furniture?

Proposed by Sambuddha Chattopadhyay

Solution. $\boxed{16}$

We work backwards. When we hit an odd number, we add 1, and if we hit an even number, we divide by 2. Starting at 2017, we get 2017, 2018, 1009, 1010, 505, 506, 253, 254, 127, 128, 64, 32, 16, 8, 4, 2, 1. Doing this process takes $\boxed{16}$ cakes in total. □

5 For all positive integers n, let the multiplicative average of n be the geometric mean of all the positive divisors of n. (The geometric mean of positive reals $x_1, x_2, \ldots, x_k$ is $\sqrt[k]{x_1 \cdot x_2 \cdots x_k}$). Let S be the number of positive integers n such that the multiplicative average of n is less than or equal to 2017. Find the remainder when S is divided by 1000.

Proposed by David Wu

Solution. $\boxed{289}$

The product of the factors of n is $n^{\tau(n)/2}$, where $\tau(n)$ denotes the number of positive divisors of n. Hence, the multiplicative average of n is always $(n^{\tau(n)/2})^{1/\tau(n)} = \sqrt{n}$. Then $\sqrt{n} \leq 2017$, so there are 2017^2 values of n which work. Modulo 1000 this is $17^2 = \boxed{289}$. □

6 Let S be the set of all positive integers less than or equal to 100. Guang randomly chooses two not-necessarily distinct elements of S and finds their greatest common divisor, d. What is the probability that d has exactly 12 factors?

Proposed by Pratik Rathore

Solution. $\boxed{\dfrac{1}{2000}}$

Since the maximum value of any element in S is 100, $d \leq 100$. Therefore we proceed by finding the positive integers less than 100 with exactly 12 factors.

For d to have 12 factors, $d = p^2qr, p^3q^2, p^5q$, or p^{11}, where p and q are primes. In the first case, $d \in \{2^2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 7, 3^2 \cdot 2 \cdot 5\} = \{60, 84, 90\}$. In the second case, $d \in \{2^3 \cdot 3^2\} = \{72\}$. In the third case $d \in \{2^5 \cdot 3\} = \{96\}$. There are no values less than 100 for the second case, since $2^{11} = 2048 > 100$.

Since every possible value for $d > 50$, the only pair of numbers associated with a certain value for d is (d, d).

Therefore, there are 5 successful outcomes, implying that the probability is $\frac{5}{100^2} = \boxed{\dfrac{1}{2000}}$. □

7 Let the set S contain all ordered triples of positive integers (x, y, z) satisfying

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{2018}{2017}$$

Compute the remainder when

$$\sum_{(x,y,z) \in S} x + y + z$$

is divided by 1000. In other words, find the sum of all $x + y + z$ over the ordered triples in S, and find the remainder when this value is divided by 1000.

Proposed by Dilhan Salgado

Solution. $\boxed{220}$

First, we should find all un-ordered triples that work. WLOG $x \leq y \leq z$. If $x \geq 3$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is too small. If $x = 2, y = 3$, then $z = 6$ gives us 1, too small, and $z = 5$ is $31/30$ which is too big, so smaller z won't work. If $x = 2$, $y \geq 4$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 1$, which won't work. Therefore, we have $x = y = 2$, $z = 2017$ or $x = 1$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{2017} \implies 2017(y+z) = yz$. As 2017 is prime, this means that 2017 divides y or z. In this case, ignore the earlier $y \leq z$ and replace it with 2017 divides y. We can write this as $y = 2017k$, so $2017(2017k + z) = 2017kz \implies 2017k + z = kz \implies z = \frac{2017k}{k-1} = 2017 + \frac{2017}{k-1}$. As k is positive, this means $k = 2$ or 2018, which implies $z = 4034$ and 2018 respectively. This gives us the unordered solutions $(x, y, z) = (2, 2, 2017), (1, 4034, 4034)$, and $(1, 2018, 2017 \cdot 2018)$ which have 3, 3, and 6 ordered solutions each. The answer is then $3(2 + 2 + 17) + 3(1 + 34 + 34) + 6(1 + 18 + 17 \cdot 18) \pmod{1000}$, which is just $\boxed{220}$. □

8 Shane has an infinite set of cards labeled $1, 1, 26, 26, 676, 676, \ldots, 26^k, 26^k, \ldots$ for all non-negative integers k. In other words, he has two cards for each integer power of 26. Shane chooses a non-empty set of cards and finds the sum of the numbers on the cards that he has selected. Let the set S contain all of the distinct sums that Shane can make. What is the remainder when the 2017th smallest element in S divided by 1000?

Proposed by Pratik Rathore

Solution. $\boxed{809}$

Consider the sums that Shane can make in base 26. Each digit of the number in base 26 will have to be 0, 1, or 2. This motivates us to think of the sums in base 3.

It is quite easy to see that if $a > b$ in base 3, then $a > b$ in base 26. Therefore, if we convert 2017 (base 10) to base 3, and then use the digits of the base 3 number in base 26, we will nearly have the answer.

$2017_{10} = 2202201_3$, so we find that the number in question is 2202201_{26}. Now we use the Chinese Remainder Theorem to find the remainder when 2202201_{26} is divided by 1000. One can compute that 2202201_{26} is $59 \pmod{125}$ and $1 \pmod{8}$. From these two congruences, we can find the answer to be $\boxed{809}$.

Remark. Just in case you're interested, the actual number is 641630809.

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