Team Number _____

MBMT Pascal Guts Round — Set 1

April 1, 2017

___________ **1** **[3]** Find $291 + 503 - 91 + 492 - 103 - 392$.

Proposed by Guang Cui

Solution. $\boxed{700}$

$291+503-91+492-103-392 = (291-91)+(503-103)+(492-392) = 200+400+100 = \boxed{700}$. □

___________ **2** **[3]** In the recording studio, Kanye has 10 different beats, 9 different manuscripts, and 8 different samples. If he must choose 1 beat, 1 manuscript, and 1 sample for his new song, how many selections can he make?

Proposed by Pratik Rathore

Solution. $\boxed{720}$

By the Fundamental Counting Principle, the answer is $10 \cdot 9 \cdot 8 = \boxed{720}$. □

___________ **3** **[3]** Pratik has a 6 sided die with the numbers 1, 2, 3, 4, 6, and 12 on the faces. He rolls the die twice and records the two numbers that turn up on top. What is the probability that the product of the two numbers is less than or equal to 12?

Proposed by Annie Zhao

Solution. $\boxed{\frac{7}{12}}$

There are 36 possibilities for the outcome of the two rolls. If he rolls a 1 on the first roll, all 6 possibilities satisfy the condition. If he rolls a 2 on the first roll, only 5 possibilities satisfy the condition. Since the numbers are factors of 12, we can easily find the number of possibilities for all other first rolls. Summing all cases, we get $\frac{6+5+4+3+2+1}{36} = \boxed{\frac{7}{12}}$. □

___________ **4** **[3]** Adam and Becky are building a house. Becky works twice as fast as Adam does, and they both work at constant speeds for the same amount of time each day. They plan to finish building in 6 days. However, after 2 days, their friend Charlie also helps with building the house. Because of this, they finish building in just 5 days. What fraction of the house did Adam build?

Proposed by Jyotsna Rao

Solution. $\boxed{\dfrac{5}{18}}$

Since Adam and Becky work at constant speeds and were planning to finish the house in 6 days, they together built 1/6 of the house together each day. They worked on the house for 5 days, so they built 5/6 of the house. Since Becky works twice as fast as Adam, Adam built $\frac{1}{3} \cdot \frac{5}{6} = \boxed{\dfrac{5}{18}}$ of the house. □

______________ 5 **[3]** Find the two-digit number such that the sum of its digits is twice the product of its digits.

Proposed by David Wu

Solution. $\boxed{11}$

Let the two digit number be $\overline{ab}$. Then $a + b = 2ab \implies 4ab - 2a - 2b = 0 \implies (2a-1)(2b-1) = 1 \implies a = 1, b = 1$. The number is therefore $\boxed{11}$. □

Team Number _____

MBMT Pascal Guts Round — Set 2

April 1, 2017

____________ **6** **[4]** There is a strange random number generator which always returns a positive integer between 1 and 7500, inclusive. Half of the time, it returns a uniformly random positive integer multiple of 25, and the other half of the time, it returns a uniformly random positive integer that isn't a multiple of 25. What is the probability that a number returned from the generator is a multiple of 30?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{7}{72}}$

If the number is a multiple of 25, then there is a $1/6$ chance, since it must be divisible by 2 and 3 as well. If it isn't, then there is a $1/6 \cdot 1/6$ chance since there is a $4/24 = 1/6$ chance that the number is divisible by 5, given that it isn't divisible by 25. The answer is $1/12 + 1/72 = \boxed{\dfrac{7}{72}}$. □

____________ **7** **[4]** Julia is shopping for clothes. She finds T different tops and S different skirts that she likes, where $T \geq S > 0$. Julia can either get one top and one skirt, just one top, or just one skirt. If there are 50 ways in which she can make her choice, what is $T - S$?

Proposed by Jyotsna Rao

Solution. $\boxed{14}$

Julia can get one top and one skirt in TS ways, just a top in T ways, and just a skirt in S ways. So, $TS + T + S = 50$, so by SFFT $(T + 1)(S + 1) = 51$. We find that one of these factors has to be 17 and the other has to be 3 for T and S to be positive integers. Since $T \geq S$, we know that $T + 1$ must be 17 and $S + 1$ must be 3. So, $T = 16, S = 2$, and $T - S = \boxed{14}$. □

____________ **8** **[4]** In cyclic quadrilateral $ABCD$, $\angle ABD = 40°$, and $\angle DAC = 40°$. Compute the measure of $\angle ADC$ in degrees. (In cyclic quadrilaterals, opposite angles sum up to 180°.)

Proposed by David Wu

Solution. $\boxed{100}$

Since $ABCD$ is cyclic, we can inscribe it in a circle. $\angle ABD \cong \angle ACD$ since they are inscribed angles sharing the same arc. Then triangle ADC is isosceles, so $m\angle ADC = 180 - 2 \cdot 40 = \boxed{100°}$. □

______________ **9** **[4]** 4 positive integers form an geometric sequence. The sum of the 4 numbers is 255, and the average of the second and the fourth number is 102. What is the smallest number in the sequence?

Proposed by Annie Zhao

Solution. $\boxed{3}$

Let the 4 positive integers be a, ar, ar^2, and ar^3. $\frac{a(r^4-1)}{r-1} = a(r+1)(r^2+1) = 255$. Since $204 = ar + ar^3 = ar(r^2+1)$, we have $\frac{r+1}{r} = \frac{a(r+1)(r^2+1)}{ar(r^2+1)} = \frac{255}{204} = \frac{5}{4}$. Solving gives $r = 4$. Substituting, we get $a \cdot 5 \cdot 17 = 255$, so $a = \boxed{3}$. □

______________ **10** **[4]** Let S be the set of all positive integers which have three digits when written in base 2016 and two digits when written in base 2017. Find the size of S.

Proposed by Pratik Rathore

Solution. $\boxed{4033}$

Three digits in base 2016 means n is from 2016^2 to $2016^3 - 1$. Two digits in base 2017 means n is from 2017 to $2017^2 - 1$.

Note that $2017 < 2016^2 < 2017^2 - 1 < 2016^3 - 1$.

Therefore S consists of all positive integers from 2016^2 to $2017^2 - 1$. The size of this set is $(2017^2 - 1) - (2016^2) + 1 = 2017^2 - 2016^2 = (2017 + 2016)(2017 - 2016) = \boxed{4033}$. □

Team Number _____

MBMT Pascal Guts Round — Set 3

April 1, 2017

___________ **11** **[5]** Find all possible values of c in the following system of equations:

$$a^2 + ab + c^2 = 31$$
$$b^2 + ab - c^2 = 18$$
$$a^2 - b^2 = 7.$$

Proposed by David Wu

Solution. $\boxed{\pm\sqrt{3}}$

Adding the former two equations, we have $a^2 + 2ab + b^2 = 49$, so $a + b = \pm 7$. Thus $a - b = \pm 1$. Then $2a = \pm 8$, so $a = \pm 4, b = \pm 3$. Substituting yields $a^2 + ab + c^2 = 16 + 12 + c^2 = 31$, whence $c^2 = 3$, so $c = \boxed{\pm\sqrt{3}}$.

Alternately, take the difference of first two equations to get $a^2 - b^2 + 2c^2 = 13$. So $2c^2 = 13 - 7 = 6 \implies c^2 = 3$. Then $c = \boxed{\pm\sqrt{3}}$. □

___________ **12** **[5]** In square $ABCD$ with side length 13, point E lies on segment CD. Segment AE divides $ABCD$ into triangle ADE and quadrilateral $ABCE$. If the ratio of the area of ADE to the area of $ABCE$ is $4 : 11$, what is the ratio of the perimeter of ADE to the perimeter of $ABCE$?

Proposed by Eric Lu

Solution. $\boxed{20 : 27}$

The triangle's area is $4/15$ of the square's. Since the square's height and the triangle's height are the same, the triangle's base is $8/15$ of the square's. Since we are dealing with ratios, let the square's side length be 15. The triangle's sides legs are 8 and 15, making the hypotenuse 17. The quadrilateral's side lengths are then 15, 15, 7, 17. The ratio is $40 : 54$ or $\boxed{20 : 27}$. □

___________ **13** **[5]** Thomas has two distinct chocolate bars. One of them is 1 by 5 and the other one is 1 by 3. If he can only eat a single 1 by 1 piece off of either the leftmost side or the rightmost side of either bar at a time, how many different ways can he eat the two bars?

Proposed by Guang Cui

Solution. $\boxed{3584}$

Let the two chocolate bars be called bar A and bar B. Thomas can choose the chocolate bars by ordering the letters in the string $AAAAABBB$, for $\binom{8}{3}$ possible orderings. Every time he takes a bar with more than 1 piece left, he can eat from either the left or right side of the bar. Thus we have to multiply by $2^4 \cdot 2^2$. The final answer is $\binom{8}{3} \cdot 2^4 \cdot 2^2 = \boxed{3584}$. □

______________ **14** **[5]** In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. The entire triangle is revolved about side BC. What is the volume of the swept out region?

Proposed by David Wu

Solution. $\boxed{672\pi}$

The volume is two cones glued together at their base. The radii of the cones is the A-altitude, 12. Hence the volume is given by $\frac{1}{3} \cdot \pi \cdot 12^2 \cdot 14 = \boxed{672\pi}$. □

______________ **15** **[5]** Find the number of ordered pairs of positive integers (a, b) that satisfy the equation $a(a-1) + 2ab + b(b-1) = 600$.

Proposed by Guang Cui

Solution. $\boxed{24}$

$a(a-1)+2ab+b(b-1) = a^2-a+b^2-b+2ab = a^2+2ab+b^2-(a+b) = (a+b)^2-(a+b) = (a+b)(a+b-1) = 600$. So $a+b = 25$. Note that $a = 1, 2, ..., 24$ will work so there are $\boxed{24}$ solutions. □

Team Number _____

MBMT Pascal Guts Round — Set 4

April 1, 2017

___________ **16** **[7]** Compute the sum of the digits of $(10^{2017} - 1)^2$.

Proposed by Kevin Qian

Solution. $\boxed{18153}$

Suppose there are n digits in $9 \cdots 9$. Squaring this gives a number with $n-1$ 9's, followed by an 8, followed by $n-1$ 0's, followed by a 1. This means the sum of the digits if always $9(n-1)+8+1 = 9n$. In particular, when $n = 2017$, the answer is $9 \cdot 2017 = \boxed{18153}$. □

___________ **17** **[7]** A right triangle with area 210 is inscribed within a semicircle, with its hypotenuse coinciding with the diameter of the semicircle. 2 semicircles are constructed (facing outwards) with the legs of the triangle as their diameters. What is the area inside the 2 semicircles but outside the first semicircle?

Proposed by Steven Qu

Solution. $\boxed{210}$

The answer is the sum of the areas of the two smaller semicircles + the area of the triangle − the area of the large semicircle. However, the sum of the areas is equal to the area of the large semicircle (Pythagorean theorem), so the answer is just $\boxed{210}$. □

___________ **18** **[7]** Find the smallest positive integer n such that exactly $\frac{1}{10}$ of its positive divisors are perfect squares.

Proposed by Guang Cui

Solution. $\boxed{5040}$

Let $n = \prod p_k^{e_k}$. If e_k is odd, then $1/2$ will work. if e_k is even, then $\frac{e_k/2+1}{e_k+1}$ of them will work, since the exponent of each must be even. Therefore, we multiply some of the fractions in $1/2, 2/3, 3/5, 4/7, 5/9, 6/11$, etc. and get $1/10$.

Note that $3/5 \cdot 2/3 \cdot 1/2 \cdot 1/2 = 1/10$, and this gives $2^4 * 3^2 * 5 * 7 = \boxed{5040}$. To prove that this is the smallest, first notice that anything beyond 7/13 cannot be used (too big). 6/11, 7/13, 4/7, 5/9 are all bad because something else is needed to cancel the 11,13,7,9.

We need at least one $1/2$ and one $3/5$ for the denominator to be 10. That gives $3/10$, so we need a $2/3$ to cancel the 3, and another $1/2$ gives $1/10$. □

19 **[7]** One day, Sambuddha and Jamie decide to have a tower building competition using oranges of radius 1 inch. Each player begins with 14 oranges. Jamie builds his tower by making a 3 by 3 base, placing a 2 by 2 square on top, and placing the last orange at the very top. However, Sambuddha is very hungry and eats 4 of his oranges. With his remaining 10 oranges, he builds a similar tower, forming an equilateral triangle with 3 oranges on each side, placing another equilateral triangle with 2 oranges on each side on top, and placing the last orange at the very top. What is the positive difference between the heights of these two towers?

Proposed by Annie Zhao

Solution. $\boxed{\frac{2}{3}(2\sqrt{6}-3\sqrt{2})}$

Connecting the centers of the oranges in Jamie's tower forms a square pyramid with base side length 4 and leg length 4. Therefore, the height of the square pyramid is $2\sqrt{2}$. The total height of the tower is $2\sqrt{2}+2$. Similarly, by connecting the centers of the oranges, Sambuddha's tower forms a tetrahedron with side length 4, so the height of the tetrahedron $4\sqrt{\frac{2}{3}}$. The total height of his tower is $4\sqrt{\frac{2}{3}}+2$. Therefore, the difference is $\boxed{\frac{2}{3}(2\sqrt{6}-3\sqrt{2})}$. □

20 **[7]** Let r, s, and t be the roots of the polynomial $x^3-9x+42$. Compute the value of $(rs)^3+(st)^3+(tr)^3$.

Proposed by Pratik Rathore

Solution. $\boxed{4563}$

Note that if k is a root of $x^3-9x+42$, then $k^3=9k-42$. We can rewrite the expression we want as $r^3s^3+s^3t^3+t^3r^3=(9r-42)(9s-42)+(9s-42)(9t-42)+(9t-42)(9r-42)=81(rs+st+tr)-756(r+s+t)+5292$. From Vieta's we have that $rs+st+tr=-9$ and $r+s+t=0$. Thus the answer is $81(-9)-756(0)+5292=\boxed{4563}$. □

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MBMT Pascal Guts Round — Set 5

April 1, 2017

___________ **21** **[9]** For all integers $k > 1$,

$$\sum_{n=0}^{\infty} k^{-n} = \frac{k}{k-1}$$

There exists a sequence of integers $j_0, j_1, ...$ such that

$$\sum_{n=0}^{\infty} j_n k^{-n} = \left(\frac{k}{k-1}\right)^3$$

for all integers $k > 1$. Find j_{10}.

Proposed by Jacob Stavrianos

Solution. $\boxed{66}$

First, we do the $\left(\frac{k}{k-1}\right)^2$ case, calling these coefficients i_n. To find this, we can take

$$\left(\frac{k}{k-1}\right)^2 = \left(1 + \frac{1}{k} + \frac{1}{k^2} + \cdots\right)\left(1 + \frac{1}{k} + \frac{1}{k^2} + \cdots\right).$$

For each term less than or equal to $1/k^n$ in the first series, it pairs with exactly one term in the second series to form a $1/k^n$ term. From this, we find there are exactly n+1 i_n terms generated (from 0 to n), each with coefficient 1, so $i_n = n + 1$.

More generally, if some x_n sequence sums to $\left(\frac{k}{k-1}\right)^z$, then a sequence $y_n = x_0 + x_1 + ...x_n$ will sum to $\left(\frac{k}{k-1}\right)^{z+1}$. In simpler terms, this is the hockey-stick identity applied to the exponent of the $\frac{k}{k-1}$. Noticing this, we find these are all just combinatoric expressions, with

$$\sum_{n=0}^{\infty} \frac{\binom{n+x}{x}}{k^n} = \left(\frac{k}{k-1}\right)^{x+1}$$

Now we just plug in the formula for $x = 2$, getting $\binom{12}{2} = 66$. Alternatively, derive the exponent 3 terms from the exponent 2 ones, getting

$$j_{10} = \sum_{n=0}^{10} i_n = 1 + 2 + 3 + 4 + \ldots + 11 = \boxed{66}$$

Remark. This complicated argument boils down to finding the number of k^{-10} terms in the expansion of

$$\left(\frac{k}{k-1}\right)^3 = \left(1+\frac{1}{k}+\frac{1}{k^2}+\cdots\right)\left(1+\frac{1}{k}+\frac{1}{k^2}+\cdots\right)\left(1+\frac{1}{k}+\frac{1}{k^2}+\cdots\right)$$

which is the same as counting the number of ordered triples of non-negative integers (a, b, c) such that $a+b+c=10$. This is just $\binom{10+2}{2} = \boxed{66}$ by stars-and-bars.

□

______ **22** **[9]** Nimi is a triangle with vertices located at $(-1,6)$, $(6,3)$, and $(7,9)$. His center of mass is tied to his owner, who is asleep at $(0,0)$, using a rod. Nimi is capable of spinning around his center of mass and revolving about his owner. What is the maximum area that Nimi can sweep through?

Proposed by Kevin Qian

Solution. $\boxed{40\pi\sqrt{13}}$

Let d be the distance from the center of mass to the origin and r be the furthest point of the triangle from the center of mass. Then, the area Nimi can cover is the region between two concentric circles of radius $d+r$ and $d-r$, which is $(d+r)^2\pi - (d-r)^2\pi = 4dr\pi$.

Now, it remains to calculate d and r. The center of mass is $\left(\frac{-1+6+7}{3}, \frac{6+3+9}{3}\right) = (4,6)$, so $d = 2\sqrt{13}$. The furthest point from the center of mass is $(-1,6)$, which is 5 units away so $r=5$. Hence, the answer is $4\cdot(2\sqrt{13})\cdot 5\cdot\pi = \boxed{40\pi\sqrt{13}}$. □

______ **23** **[9]** The polynomial $x^{19}-x-2$ has 19 distinct roots. Let these roots be $\alpha_1, \alpha_2, \ldots, \alpha_{19}$. Find $\alpha_1^{37}+\alpha_2^{37}+\cdots+\alpha_{19}^{37}$.

Proposed by Daniel Zhu

Solution. $\boxed{74}$

Note that for all i, $\alpha_i^{19} = \alpha_i + 2$, so $\alpha_i^{19+k} = \alpha_i^{k+1} + 2\alpha_i^k$.

Then

$$\begin{aligned}
\alpha_1^{37}+\alpha_2^{37}+...+\alpha_{19}^{37} &= (\alpha_1^{19}+2\alpha_1^{18}) + (\alpha_2^{19}+2\alpha_2^{18}) + ... + (\alpha_{19}^{19}+2\alpha_{19}^{18}) \\
&= (\alpha_1+4+4/\alpha_1) + (\alpha_2+4+4/\alpha_2) + ... + (\alpha_{19}+4+4/\alpha_{19}) \\
&= (\alpha_1+\alpha_2+...+\alpha_{19}) + 4\cdot 19 + 4\cdot(1/\alpha_1+1/\alpha_2+...+1/\alpha_{19}) \\
&= 0+76+4\cdot(-1/2) \\
&= \boxed{74}.
\end{aligned}$$

□

____________ **24** **[9]** I start with a positive integer n. Every turn, if n is even, I replace n with $\frac{n}{2}$, otherwise I replace n with $n-1$. Let k be the most turns required for a number $n < 500$ to be reduced to 1. How many values of $n < 500$ require k turns to be reduced to 1?

Proposed by Steven Qu

Solution. $\boxed{4}$

Consider n in binary. Then, every turn, an ending 1 becomes a 0, and an ending 0 disappears. This means that a 1 takes 2 turns to get rid of and a 0 takes 1 turn (we want more 1's). 511 is 111111111_2, so $101111111_2, 110111111_2, 111011111_2$, and 111101111_2 are optimal (111110111_2 is too big). The answer is therefore $\boxed{4}$. □

____________ **25** **[9]** In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let I and O be the incircle and circumcircle of ABC, respectively. The altitude from A intersects I at points P and Q, and O at point R, such that Q lies between P and R. Find PR.

Proposed by Steven Qu

Solution. $\boxed{\dfrac{31}{4} + \sqrt{15}}$

Let M be the intersection of AB and BC. D, E, F be OI tangent parts. Drop perpendicular from I to PQ and denote the intersection as J. We know $IJ = MD = 1$, $ID = IQ$, $r = 4$, so $IP = IQ = \sqrt{15}$ and $PQ = 2\sqrt{15}$. Applying Power of a Point, $AP \cdot AQ = AE^2 = 49, AP = 8 - \sqrt{15}$ so $PM = 4 + \sqrt{15}$. By similar triangles, $MR = 15/4$ and $PR = \boxed{\dfrac{31}{4} + \sqrt{15}}$.

https://lh6.googleusercontent.com/-2bSvB_JHp2s/V4_R06jvoYI/AAAAAAAABIs/xK2x2uRKUeQe1u17l6DBnBgCL0B/w768-h1024-no/2016-07-20.png □

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MBMT Pascal Guts Round — Set 6

April 1, 2017

___________ **26** **[12]** Submit a decimal n to the nearest thousandth between 0 and 200. Your score will be $\min(12, S)$, where S is the non-negative difference between n and the largest number less than or equal to n chosen by another team (if you choose the smallest number, $S = n$). For example, 1.414 is an acceptable answer, while $\sqrt{2}$ and 1.4142 are not.

Proposed by Guang Cui

Solution. $\boxed{\text{N/A}}$

This is similar to Reaper. □

___________ **27** **[12]** Guang is going hard on his YNA project. From 1:00 AM Saturday to 1:00 AM Sunday, the probability that he is not finished with his project x hours after 1:00 AM on Saturday is $\frac{1}{x+1}$. If Guang does not finish by 1:00 AM on Sunday, he will stop procrastinating and finish the project immediately. Find the expected number of minutes A it will take for him to finish his project.

An estimate of E will earn $12 \cdot 2^{-|E-A|/60}$ points.

Proposed by David Wu

Solution. $\boxed{\approx 193.132549}$

The probability that Guang will not finish after x hours is $\frac{1}{x+1}$. Then the probability that he completes at time x is $dx = -\frac{1}{(x+1)^2}$. We can split it up into two cases: Either Guang finishes within the 24 hours, or he finishes it immediately at the end. The first expected value is

$$\int_0^{24} -\frac{x}{(x+1)^2}dx.$$

This evaluates to $\ln(x+1) + \frac{1}{x+1}\Big|_0^{24} = \ln(25) - \ln(1) + 1/25 - 1 = \ln(25) - 24/25$. The second expected value: It takes Guang 24 hours to complete, with probability $1/25$. Thus, the final answer is $60(\ln(25) - 24/25 + 24/25) = 60\ln(25)$. This is about 193.132549. □

___________ **28** **[12]** All the diagonals of a regular 100-gon (a regular polygon with 100 sides) are drawn. Let A be the number of distinct intersection points between all the diagonals. Find A.

An estimate of E will earn $12 \cdot (16\log_{10}(\max(\frac{E}{A}, \frac{A}{E})) + 1)^{-\frac{1}{2}}$ or 0 points if this expression is undefined.

Proposed by David Wu

Solution. $\boxed{3731201}$

A reasonable upper bound is $\binom{100}{4}$ since every four points determine one quadrilateral and therefore one intersection of diagonals. However, there are points through which more than two diagonals pass. Thus, the answer is smaller than $\binom{100}{4}$, but not by a large amount. $\binom{100}{4} \approx 3.9 \cdot 10^6$, so an answer between $3.4 \cdot 10^6$ and $3.9 \cdot 10^6$ would be reasonable. The real answer can be found here: `http://www.wolframalpha.com/input/?i=diagonals+of+100-gon`. □

____________ **29** **[12]** Find the smallest positive integer A such that the following is true: if every integer $1, 2, \ldots, A$ is colored either red or blue, then no matter how they are colored, there are always 6 integers among them forming an increasing arithmetic progression that are all colored the same color.

An estimate of E will earn $12\min(\frac{E}{A}, \frac{A}{E})$ points or 0 points if this expression is undefined.

Proposed by Guang Cui

Solution. $\boxed{1132}$

`http://www.tandfonline.com/doi/abs/10.1080/10586458.2008.10129025` □

____________ **30** **[12]** For all integers $n \geq 2$, let $f(n)$ denote the smallest prime factor of n. Find

$$A = \sum_{n=2}^{10^6} f(n).$$

In other words, take the smallest prime factor of every integer from 2 to 10^6 and sum them all up to get A.

You may find the following values helpful: there are 78498 primes below 10^6, 9592 primes below 10^5, 1229 primes below 10^4, and 168 primes below 10^3.

An estimate of E will earn $\max(0, 12 - 4\log_{10}(\max(\frac{E}{A}, \frac{A}{E})))$ or 0 points if this expression is undefined.

Proposed by David Wu

Solution. $\boxed{37568404989}$

Getting a handle on the order of magnitude of the answer shouldn't be too bad. For example, half of the numbers are divisible by 2. For simplicity consider $f(1)$ to be in the sum even though it doesn't contribute to the sum in any way. A first order approximation yields $10^6(2 \cdot 1/2 + 3 \cdot 1/2 \cdot 1/3 + 5 \cdot 1/2 \cdot 2/3 \cdot 1/5 + \ldots)$ since for a prime p_k there are approximately

$$\prod_{i=1}^{k-1}\left(1 - \frac{1}{p_i}\right) \cdot \frac{1}{p_k}$$

numbers whose smallest prime factor is p_k.

Intuitively, these small primes should not contribute too much to the sum. Note that if n has a prime factor above 10^3, then it must be prime. Thus we first try estimate the terms up to the largest prime below 10^3.

Verifying by hand, the first few terms in the parentheses evaluate to $1 + 1/2 + 1/3 + 4/15 + 8/35 + 16/77 + ...$ Note however, that we can fudge some factors around get that the average term will be smaller than 0.2 after 8/35. So we get an upper bound of $10^6(2 + 1/5 \cdot 160) = 3.4 \cdot 10^7$.

Now we need to essentially calculate sum of primes p where $10^3 < p < 10^6$. This essentially boils down to estimating the average prime between 10^3 and 10^6. Just by looking at the provided numbers, we can tell that most of the primes are between 10^5 and 10^6. Thus, we can reasonably estimate the average prime in our given interval to be $5 \cdot 10^5$. The given information also tells us there are roughly 80000 primes between 10^3 and 10^6. Thus, these primes contribute around $8 \cdot 10^4 \times 5 \cdot 10^5 = 4 \cdot 10^{10}$ to our sum. We now see that our original terms are insignificant. Our estimate of $4 \cdot 10^{10}$ is remarkably close to the actual answer, $3.8 \cdot 10^{10}$.

$\square$