MBMT Geometry Round — Pascal

April 1, 2017

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered by fewer competitors will be weighted more heavily. Please write your answers in the simplest possible form.

1 Angle X has a degree measure of 35 degrees. What is the supplement of the complement of angle X?

The complement of an angle is 90 degrees minus the angle measure. The supplement of an angle is 180 degrees minus the angle measure.

Proposed by David Wu

Solution. $\boxed{125^\circ}$

The complement of X is $90 - 35 = 55^\circ$. The supplement of 55° is $180 - 55 = \boxed{125^\circ}$. □

2 A car that always travels in a straight line starts at the origin and goes towards the point $(8, 12)$. The car stops halfway on its path, turns around, and returns back towards the origin. The car again stops halfway on its return. What are the car's final coordinates?

Proposed by David Wu

Solution. $\boxed{(2, 3)}$

We use the midpoint formula twice to solve the problem. On the first path, the car stops at $(\frac{0+8}{2}, \frac{0+12}{2}) = (4, 6)$. On the path back towards the origin, the car stops at $(\frac{4+0}{2}, \frac{6+0}{2}) = \boxed{(2, 3)}$, which is the final answer. □

3 Let ABC be an isosceles triangle such that $AB = BC$ and all of its angles have integer degree measures. Two lines, ℓ_1 and ℓ_2, trisect $\angle ABC$. ℓ_1 and ℓ_2 intersect AC at points D and E respectively, such that D is between A and E. What is the smallest possible integer degree measure of $\angle BDC$?

Proposed by Pratik Rathore

Solution. $\boxed{61^\circ}$

Let x be the measure of $\angle ABC$. From angle chasing, we can find that $\angle BDC = 90^\circ - x/6$. Clearly $0 < x < 180$. Then $60 < \angle BDC < 90$. Therefore the minimum is $\boxed{61^\circ}$.

Remark. This can be attained when $\angle ABC = 174^\circ$ and the other two angles are 3 degrees.

□

4 In rectangle $ABCD$, $AB = 9$ and $BC = 8$. W, X, Y, and Z are on sides AB, BC, CD, and DA, respectively, such that $AW = 2WB$, $CX = 3BX$, $CY = 2DY$, and $AZ = DZ$. If WY and XZ intersect at O, find the area of $OWBX$.

Proposed by Guang Cui

Solution. $\boxed{\dfrac{93}{11}}$

Use coordinates. Setting B as the origin is most convenient. Then, $X = (0, -2), W = (-3, 0), A = (-9, 0)$, etc. The equation for line XZ is therefore $y = \frac{2x}{9} - 2$ and the equation for line WY is $y = \frac{8}{3}(x + 3)$ or $y = \frac{8x}{3} + 8$. Setting these equal, $x(2/9 - 24/9) = 10$, so $x = -45/11$ and $y = -32/11$. The area is straightforward by Shoelace or by finding the area of the circumscribing rectangle and subtracting off the areas of the two right triangles. We get $\dfrac{90/11 + 96/11}{2} = \boxed{\dfrac{93}{11}}$. □

5 Consider a regular n-gon with vertices $A_1A_2 \ldots A_n$. Find the smallest value of n so that there exist positive integers $i, j, k \leq n$ with $\angle A_iA_jA_k = \dfrac{34^\circ}{5}$.

Proposed by Pratik Rathore

Solution. $\boxed{450}$

We inscribe the n-gon inside a circle. From the inscribed angle theorem, it is evident that $\angle A_iA_jA_k = \widehat{A_iA_k}/2$. The measure of $\widehat{A_iA_k}$ is $360m/n$ for some positive integer m. So we have the equation $180m/n = 34/5$ for some integral m and n. Thus $m/n = 34/900 = 17/450$. Since 17 and 450 have no common factor greater than 1, we conclude that the minimum value for n is $\boxed{450}$.

Remark. For $n = 450$, a working configuration is $\angle A_1A_{19}A_{18} = \dfrac{34^\circ}{5}$.

□

6 In right triangle ABC with $\angle A = 90^\circ$ and $AB < AC$, D is the foot of the altitude from A to BC, and M is the midpoint of BC. Given that $AM = 13$ and $AD = 5$, what is $\frac{AB}{AC}$?

Proposed by David Wu

Solution. $\boxed{\dfrac{1}{5}}$

Since $\triangle ABC$ is a right triangle, M is the circumcenter of $\triangle ABC$, implying that $AM = BM = CM = 13$. Since $\triangle ADM$ is a right triangle with hypotenuse 13 and leg 5, the other leg has length 12. Then $DC = 25$, so $\dfrac{AB}{AC} = \dfrac{AD}{DC} = \boxed{\dfrac{1}{5}}$. □

7 An ant is on the circumference of the base of a cone with radius 2 and slant height 6. It crawls to the vertex of the cone X in an infinite series of steps. In each step, if the ant is at a point P, it crawls along the shortest path on the exterior of the cone to a point Q on the opposite side of the cone such that $2QX = PX$. What is the total distance that the ant travels along the exterior of the cone?

Proposed by Annie Zhao

Solution. $\boxed{6\sqrt{3}}$

The cone is constructed from a 120 degree arc. The first straight line creates a 30-60-90 triangle, so the line itself has length $3\sqrt{3}$. The length of the next segment is half of the length of the previous segment. Therefore, the total distance can be calculated as an infinite geometric series, so the answer is $\boxed{6\sqrt{3}}$. □

8 There is an infinite checkerboard with each square having side length 2. If a circle with radius 1 is dropped randomly on the checkerboard, what is the probability that the circle lies inside of exactly 3 squares?

Proposed by Guang Cui

Solution. $\boxed{1 - \dfrac{\pi}{4}}$

Notice that the circle will cover either 3 or 4 squares, unless it is dropped exactly on the center of a square or a midline, which has probability 0. We find the probability that it covers 4 squares. Assume the center of the circle lies in some square. Then, covering 4 squares is equivalent to covering one of the four corners of the square it resides in. This corresponds to a quarter circle centered at each corner, which has area π and thus probability $\frac{\pi}{4}$. Therefore, the probability of covering only 3 squares is $\boxed{1 - \dfrac{\pi}{4}}$. □