MBMT Counting and Probability Round — Pascal

April 1, 2017

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered by fewer competitors will be weighted more heavily. Please write your answers in the simplest possible form.

1 4 people are at a party, and every pair of people shakes hands with each other. How many total handshakes are there?

Proposed by Mr. Stein

Solution. $\boxed{6}$

It suffices to count the number of pairs of people. To count this, we can choose one person to be the first person in the pair, which can be done in 4 ways, and a second person who is not the first, which can be done in 3 ways. However, this way each pair is counted twice, since pairs are not ordered. Therefore, the answer is $\frac{4 \cdot 3}{2} = \boxed{6}$.

Alternatively note that the answer is $\binom{4}{2} = \boxed{6}$. □

2 A palindrome is a nonnegative integer that reads the same forwards and backwards. For example, 12321 and 0 are both palindromes. Find the number of palindromes between 1000 and 9999, inclusive.

Proposed by David Wu

Solution. $\boxed{90}$

Four-digit palindromes are of the form $\overline{abba}$, so we just need the number of ordered pairs of digits (a, b). Since $a \neq 0$, there are $9 \cdot 10$ such pairs, and the answer is $\boxed{90}$. □

3 Stan has five friends: Allen, Brian, Catherine, Daniel, and Evan. Each of the 6 people took a test and the teacher told each of them their own score privately. Now they want to share their scores with each other. In a conversation, Person A tells Person B all the scores they know, but not vice versa. What is the minimum number of conversations required for every person to know all six scores?

Proposed by Kevin Qian

Solution. $\boxed{10}$

Consider the earliest moment when someone knows everyone else's score. WLOG let this person be Stan. There must have been 5 conversations beforehand because each of the other five people must have revealed their score to someone else. On the other hand, this first person is unique, and we require at least 5 turns to inform each of the other people. Hence, the minimum is $\boxed{10}$. We can achieve 10 by having everyone tell Stan their score and have Stan distribute the information. □

4 Guang is listening to an album that consists of 21 songs. 13 of the songs are 4 minutes long, 7 of the songs are 1 minute long skits, and one song is 12 minutes long. If he shuffles the songs randomly, what is the probability that the first 3 songs played last a total of at least 9 minutes?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{73}{95}}$

Complementary counting gives the easiest solution to this problem.

There are only two cases where the first three songs last for less than 9 minutes: they are three skits, or two skits and a shorter song. There are $\binom{7}{3}$ ways for first case to occur, and $\binom{7}{2} \cdot 13$ ways for the second case to occur. Since there are $\binom{21}{3}$ ways to choose the first three songs in total, the answer is

$$1 - \frac{\binom{7}{3} + \binom{7}{2} \cdot 13}{\binom{21}{3}} = \boxed{\frac{73}{95}}$$

Remark. The album being referred to in this problem is *The College Dropout* by Kanye West.

□

5 A jar with N candies contains only mints and lollipops, all of which are distinguishable. Mindy wants to take some, but not all, of the mints. Lolly wants to take some, but not all, of the lollipops. Let M be the number of ways in which Mindy can choose her mints, and let L be the number of ways in which Lolly can choose her lollipops. If $M - L = 3840$, what is N?

Proposed by Jyotsna Rao

Solution. $\boxed{20}$

Let x be the number of mints in the jar, and let y be the number of lollipops. $M = 2^x - 2$ and $L = 2^y - 2$ because Mindy and Lolly can choose all subsets of the mints and lollipops (respectively) except the subsets containing all or none of their respective candies. So, $M - L = 2^x - 2^y = 3840 = 2^{12} - 2^8$. Therefore, $x = 12, y = 8$, and $N = x + y = \boxed{20}$. □

6 There are 10 vans in a row, and Daniel paints each of them either white or black, independently and with equal probability. What is the probability that after painting all 10 vans, there will be two vans next to each other that are both white?

Proposed by Guang Cui

Solution. $\boxed{\dfrac{55}{64}}$

We use complementary counting to solve this problem. Consider the number of countings that do not have any consecutive white vans. We can establish a Fibonacci recurrence to count this: if the first van is white, the second van must be black so there are a_{n-2} colorings, and if the first van is black then there are a_{n-1} colorings. Since $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$, we find $a_{10} = 144$. Thus the probability is $1 - \dfrac{144}{2^{10}} = \boxed{\dfrac{55}{64}}$. □

7 Let S be $\{1, 2, 3, ..., 2017\}$. How many ordered 1000-tuples of sets $(X_1, X_2, ..., X_{1000})$ are there such that $X_1 \subseteq X_2 \subseteq ... \subseteq X_{1000} \subseteq S$?

Proposed by Daniel Zhu

Solution. $\boxed{1001^{2017}}$

Consider each element of S. It can appear in

- only S
- only X_{1000}, S
- only X_{999}, X_{1000}, and S

  $\vdots$

- all the sets

Since there are 1001 possibilities, the total number of possibilities is $\boxed{1001^{2017}}$. □

8 Each point in a 4 by 6 rectangular grid of lattice points is colored either red or blue. A coloring is "good" if it does not contain 4 points of the same color that form the vertices of a rectangle with edges parallel to the grid axes. How many "good" colorings of the 4 by 6 grid are there?

Proposed by Guang Cui

Solution. $\boxed{720}$

Consider the grid as 6 columns, each with 4 points. If one column has 4 points of the same color (say red), then if another column contains two red points, there must be a rectangle, consisting of those two red points and the two red points in the column with all red points. Therefore, the other 5 columns must contain at most 1 red point, so there are two columns with at least 3 blue points each, which guarantees a blue rectangle. We arrive at a similar contradiction if a column contains 3 points of the same color (say red): any other column must contain at most 2 red points, and the two red points cannot both coincide with the 3 red points in the first column. Therefore, there are a maximum of 3 columns that can contain 2 red points, so the remaining 2 columns must contain at most 1 red point. However, this means those two columns both contain at least 3 blue points, which must result in a blue rectangle. Thus, each column of 4 must have two red and two blue points, and two columns cannot have the same configuration, since that would result in a rectangle. Any of the $\binom{4}{2}! = \boxed{720}$ permutations of the columns will work. □