MBMT Algebra Round — Pascal

April 1, 2017

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered by fewer competitors will be weighted more heavily. Please write your answers in the simplest possible form.

1 Mr. Street is using his newly patented stapling technique to staple packets for his class. If there are 1000 sheets of paper when Mr. Street starts stapling, each packet contains 5 sheets of paper, and each packet takes 2 seconds to staple, how long in seconds does it take Mr. Street to staple all the packets for his class?

Proposed by David Wu

Solution. $\boxed{400}$

Mr. Street must make $\frac{1000}{5} = 200$ packets, and each packet takes 2 seconds to staple, so the total time required is $200 \cdot 2 = \boxed{400}$. □

2 Kanye West really likes driving from his house to the studio in his Mercedes. On a day where traffic is moving smoothly, Kanye can get to the studio from his house in 30 minutes. On a day with a lot of traffic, it takes Kanye 50 minutes to reach the studio from his house, and his average speed is 20 miles per hour slower than usual. How far, in miles, is the studio from Kanye's house?

Proposed by Pratik Rathore

Solution. $\boxed{25}$

Let s be Kanye's speed in MPH. Then since $d = rt$ we have $d = \frac{s}{2} = \frac{5(s-20)}{6}$. This means that $s = 50$. Then the distance is $50/2 = \boxed{25}$. □

3 Alex is playing with some goo. The goo is very magical: each morning at 7:00 AM it will make itself 60% longer. Every day, Alex wakes up at 8:00 AM, plays with it and stretches it by another 25%. If the goo is 1 meter long at 6:59 AM on Monday, on what day of the week will it surpass 2017 meters in length?

Proposed by Kevin Qian

Solution. $\boxed{\text{Thursday}}$

We claim that at noon on day i, the goo will be 2^i meters long. We can prove this claim using induction. Notice that each day, it gets longer by a factor of $1.6 \cdot 1.25 = 2$. Furthermore, it's true for the base case of $n = 1$, so the result is true for all n. Now, we take $\lceil \log_2 2017 \rceil$ to get 11 days. Now, counting 11 days with the first day being Monday, we get the answer of $\boxed{\text{Thursday}}$. □

4 Let $P(x) = x^4 - 24x^2 + 36$ with roots a, b, c and d such that $a > b > c > d$. Find $ab + cd$.

Proposed by Jyotsna Rao

Solution. $\boxed{12}$

$P(x) = x^4 + 12x^2 + 36 - 36x^2 = (x^2+6)^2 - (6x)^2 = (x^2+6x+6)(x^2-6x+6)$. Note that the roots of x^2+6x+6 are $-3 \pm \sqrt{3}$ and the roots of x^2-6x+6 are $3 \pm \sqrt{3}$ by the quadratic formula. Thus $a = 3+\sqrt{3}, b = 3-\sqrt{3}, c = -3+\sqrt{3}, d = -3-\sqrt{3}$. The answer is thus $(3+\sqrt{3})(3-\sqrt{3}) + (-3+\sqrt{3})(-3-\sqrt{3}) = \boxed{12}$.

Alternately, since $P(x)$ is even, we have that a, b are positive and that $d = -a$ and $c = -b$. Thus we have $ab = cd$. But from Vieta's, $abcd = 36 \Longrightarrow (ab)^2 = 36 \Longrightarrow ab = 6$, since ab has to be positive. The answer is thus $6 + 6 = \boxed{12}$. □

5 Andrew has n muffins, where n is a positive integer, arranged in an a by b rectangular array. He eats all the muffins in the top row and all the muffins in the left column and then realizes that he has eaten half of the muffins. Compute n.

Proposed by Guang Cui

Solution. $\boxed{12}$

He ate $a+b-1$ muffins, so $2(a+b-1) = ab$, and by SFFT, $(a-2)(b-2) = 2$. Therefore, $a-2 = 2$ and $b-2 = 1$ (or vice versa), and $n = 3 \times 4 = \boxed{12}$. □

6 Let S be the set of points (x, y) such that

- x and y are integers
- (x, y) is not the origin
- $0 \leq x \leq 10$ and $0 \leq y \leq 10$

For every point P in S, the circle with diameter OP, where O is the origin, is drawn. What is the total area of all the circles drawn?

Proposed by Daniel Zhu

Solution. $\boxed{\dfrac{4235\pi}{2}}$

If $P = (x, y)$, then the area of the corresponding circle is $\frac{\pi}{4}(x^2+y^2)$. Hence, we want to compute

$$\sum_{x=0}^{10}\sum_{y=0}^{10}\frac{\pi}{4}(x^2+y^2)$$

$$\sum_{x=0}^{10}\sum_{y=0}^{10}\frac{\pi}{4}(x^2) + \sum_{x=0}^{10}\sum_{y=0}^{10}\frac{\pi}{4}(y^2)$$

$$11\sum_{x=0}^{10}\frac{\pi}{4}(x^2) + 11\sum_{y=0}^{10}\frac{\pi}{4}(y^2)$$

$$\frac{11\pi}{2}\sum_{x=0}^{10} x^2$$

$$\frac{11\pi}{2}\cdot\frac{10(10+1)(2\cdot 10+1)}{6}$$

$$\boxed{\frac{4235\pi}{2}}$$

□

7 For each positive integer k, Tsew Eynak writes down the number k k times, resulting in the sequence $(1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots)$. Let $f(n)$ be the average of all the numbers through the nth number in Tsew's sequence. What is the value of n for which $f(n) = \frac{46}{3}$?

Proposed by Pratik Rathore

Solution. $\boxed{264}$

Clearly as n increases, the value of $f(n)$ increases. Since the values in Tsew's list step up at triangular numbers, let us consider the value of $f(n)$ when $n = \frac{k(k+1)}{2}$.

The number of elements in the sum is just $n = \frac{k(k+1)}{2}$. The sum itself is $1^2+2^2+...+k^2 = \frac{k(k+1)(2k+1)}{6}$. Thus the mean is $\frac{k(k+1)(2k+1)/6}{k(k+1)/2} = \frac{2k+1}{3}$. We want to bound the value of 46/3 based on this result. We set up the inequality $\frac{2k+1}{3} \le \frac{46}{3} \le \frac{2k+3}{3}$, which gives that the only integral value for k is 22. All we must now do is find the number of 23's contained in Tsew's sum.

Let the number of 23's be x. We must solve

$$\frac{[(22)(22+1)(22\cdot 2+1)/6]+23x}{[(22)(22+1)/2]+x} = \frac{46}{3}$$

This is the same as $\frac{165\cdot 23+23x}{11\cdot 23+x} = \frac{46}{3} \implies 46(11\cdot 23+x) = 3(165\cdot 23+23x) \implies 23x = 506\cdot 23 - 495\cdot 23 \implies x = 11$. Thus the answer is $22\cdot 23/2+11 = \boxed{264}$. □

8 Let $f(x) = x^3+3x^2+5x+\frac{1}{2}$ have roots r_1, r_2, r_3. Compute

$$\left(r_1+r_2+\frac{1}{r_1r_2}\right)\left(r_2+r_3+\frac{1}{r_2r_3}\right)\left(r_3+r_1+\frac{1}{r_3r_1}\right)$$

Proposed by Pratik Rathore

Solution. $\boxed{-\frac{135}{2}}$

From Vieta's, $r_1r_2r_3 = -1/2$. Then $\frac{1}{r_1r_2} = -2r_3$. Thus $r_1+r_2+\frac{1}{r_1r_2} = r_1+r_2-2r_3 = r_1+r_2+r_3-3r_3$.

Vieta's also tells us that $r_1+r_2+r_3 = -3$. Therefore we can rewrite the former expression as $-3-3r_3 = -3(1+r_3)$. From symmetry, we can rewrite the original product as $(-3)^3(1+r_1)(1+r_2)(1+r_3) = (-3)^3(-1)^3(-1-r_1)(-1-r_2)(-1-r_3) = 27(-1-r_1)(-1-r_2)(-1-r_3)$. But $(-1-r_1)(-1-r_2)(-1-r_3) = f(-1) = -5/2$. Therefore the answer is $27(-5/2) = \boxed{-135/2}$. $\square$