MBMT Number Theory Round — Euler

Full Name _____

Team Number _____

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is worth the same number of points. Please write your answers in the simplest possible form.

1. What is $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$ of 1000?

2. Compute the last digit of $1! + 2! + 3! + 4! + 5! + \cdots + 10!$.

3. A science teacher is opening his students' lockers, which are in a row and are numbered 1 through 100 in order. He opens every other locker, starting from locker 1 (so he opens lockers 1, 3, 5, and so on). Then he goes back to the beginning and opens every other unopened locker (so he opens 2, 6, 10, and so on). He continues this until all lockers are open. What is the number of the last locker he opens?

4. Compute the number of zeros at the end of the base-12 representation of $30_{10}!$. Express your answer in base 10.

5. $4004 = 44 \cdot 91$, so 4004 is a multiple of the number obtained by removing its middle two digits. How many 4-digit palindromes $\underline{abba}$ are divisible by the 2-digit palindrome $\underline{aa}$?

6. Bob and his 30 friends are sharing a huge plate of cookies. When they try to split the cookies evenly, there are 17 cookies left over. In an effort to create an even split, Bob volunteers not to receive any cookies. However, Bob's 30 friends find that if they try to split the cookies evenly, there are 16 cookies left over. What is the smallest number of cookies that could be on the plate?

7. Let r be a rational number between 0 and 1, exclusive. The decimal representation of r is $0.abcabcabc\ldots$, where $b = 2(a + c)$. When expressed as a common fraction, the denominator of r is prime. Compute r as a simplified fraction.

8. Find the number of positive integers less than or equal to 400 that can be expressed as a difference of perfect squares in at least two different ways. (Note that 0 is a perfect square.) The following information may be useful:

   - There are 25 primes under 100.
   - There are 46 primes under 200.
   - There are 62 primes under 300.
   - There are 78 primes under 400.