**1.** **[4]** Spongebobert Squaretrousers lives in a cube under the sea of side length 2, and Spongerobert Rectanglepants lives in a rectangular prism above the sea with double the height, triple the length, and triple the width of Spongebobert's house. What is the volume of Spongerobert's house?

*Proposed by Thomas Yao.*

**Answer:** $\boxed{144}$

**Solution:** Spongebobert's cube initially has length 2, width 2, and height 2. After doubling the height, tripling the length, and tripling the width, the dimensions of Spongerobert's house should be length 6, width 6, and height 4. The volume of his rectangular prism is thus $6 \cdot 6 \cdot 4 = \boxed{144}$.

**2.** **[4]** Sam flips a fair coin 4 times. What is the probability that he flips the same face twice in a row, and then flips the other face twice in a row?

*Proposed by Evan Zhang.*

**Answer:** $\boxed{\frac{1}{8}}$

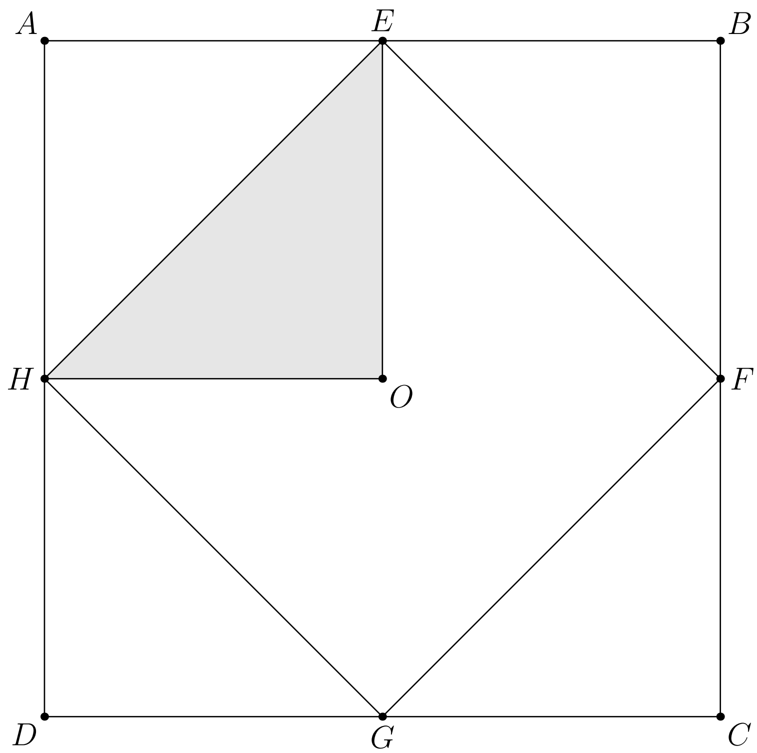
**Solution:** Evan can flip either two heads in a row and then two tails in a row, or two tails in a row and then two heads in a row, so only 2 combinations of flips. There are $2^4$ possible combinations of 4 rolls he can make, and only 2 that work, so the answer is $2/16 = \boxed{\frac{1}{8}}$.

**3.** **[4]** Square $ABCD$ has side length 8. What is the area of the square formed by the midpoints of the sides of $ABCD$?

*Proposed by Evan Zhang.*

**Answer:** $\boxed{32}$

**Solution:**




Note that $ABCD$ can be split into four smaller squares, one of which is $AEOH$. Since $\triangle EOH$ is half the area of $AEOH$, we can say that $EFGH$ is half the area of $ABCD$. Thus, the answer is $\frac{1}{2} \cdot 8^2 = \boxed{32}$.

**4.** **[5]** Let $\lfloor x \rfloor$ denote the greatest integer less than $x$. What is the unique real number $x$ that satisfies $x \cdot \lfloor x \rfloor = 10$?

*Proposed by William Roe.*

**Answer:** $\boxed{\dfrac{10}{3}}$

**Solution:** We know that

$$\lfloor x \rfloor^2 \leq x \cdot \lfloor x \rfloor \leq x^2.$$

Since, $x \cdot \lfloor x \rfloor = 10$, we have that $10 \leq x^2$, so $x \geq 3$. In the same way, $\lfloor x \rfloor^2 \leq 10$, so $\lfloor x \rfloor \leq 3$. Together, these conditions imply $\lfloor x \rfloor = 3$, so $3x = 10$ and thus $x = \boxed{\dfrac{10}{3}}$.

**5.** **[5]** Let a *half-prime* number be a positive integer $n$ such that among the first $n$ positive integers, *exactly* half of them are prime. What is the average of all half-prime numbers?

*Proposed by Siyuan Yan.*

**Answer:** $\boxed{5}$

**Solution:** We can begin by counting upwards with small $n$. For *exactly* half of the integers to be prime, $n$ must be even.

- For $n = 2$, 2 is prime, so 2 is *half-prime*.
- For $n = 4$, 2 and 3 are prime, so 4 is *half-prime*.
- For $n = 6$, 2, 3, and 5 are prime, so 6 is *half-prime*.
- For $n = 8$, 2, 3, 5, and 7 are prime, so 8 is *half-prime*.
- For $n = 10$, 2, 3, 5, and 7, are prime, so 10 is not *half-prime*.

Now, assume all of the odd numbers less than $n$ except 1 are prime. Thus, $n$ would be half-prime. However, this is clearly not true for $n \geq 10$ as 9 is not prime. So, we are done. The average is thus $\frac{1}{4}(2+4+6+8) = \frac{1}{4}(20) = \boxed{5}$.

**6.** **[5]** Alice, Bill, and Chris each start with the same amount of apple juice in their cups. Alice pours 2/3 of her apple juice into Bill's cup, and afterwards Bill pours 4/5 of his current apple juice into Chris's cup. If they pour all of their juice into one bowl, what fraction of the juice in the bowl was poured from Chris's cup?

*Proposed by Sam Easaw.*

**Answer:** $\boxed{\frac{7}{9}}$

**Solution:** Assume Alice, Bill, and Chris, have 3 ounces of juice each. Then, Alice will give $3 \cdot 2/3 = 2$ ounces to Bill, leaving Alice with 1 ounce and Bill with $3 + 2 = 5$ ounces. Then, Bill gives $5 \cdot 4/5 = 4$ ounces to Chris, leaving Bill with 1 ounce and Chris with $3 + 4 = 7$ ounces. When they pour all of the juice into one container, 7 ounces out of the $3 + 3 + 3 = 9$ ounces total will come from Chris's cup. Thus, the answer is $\boxed{\frac{7}{9}}$.

**7.** **[6]** Let $ABCDE$ be a convex pentagon with segment $\overline{BC}$ parallel to segment $\overline{AD}$, $AB = BC = CD = 4$, and $DE = EA = AD = 8$. What is the area of pentagon $ABCDE$?

*Proposed by Sam Easaw.*

**Answer:** $\boxed{28\sqrt{3}}$

**Solution:** Since $DE = EA = AD = 8$, $\triangle ADE$ is an equilateral triangle of side length 8. Now, considering quadrilateral $ABCD$, since $AB = CD$ and $BC \parallel AD$, $ABCD$ must be an isosceles trapezoid of base lengths $AD = 8$ and $BC = 4$. If we extend lines $AB$ and $CD$ to meet at a point $F$, we see that since $AD/BC = 2$, $BF = CF = BC = 4$, by similar triangles. Thus, we can express the area of $ABCDE$ as

$$[\triangle AFD] + [\triangle ADE] - [\triangle BFC].$$

These are all equilateral triangles with side lengths 8, 8, and 4, respectively, so the desired area is

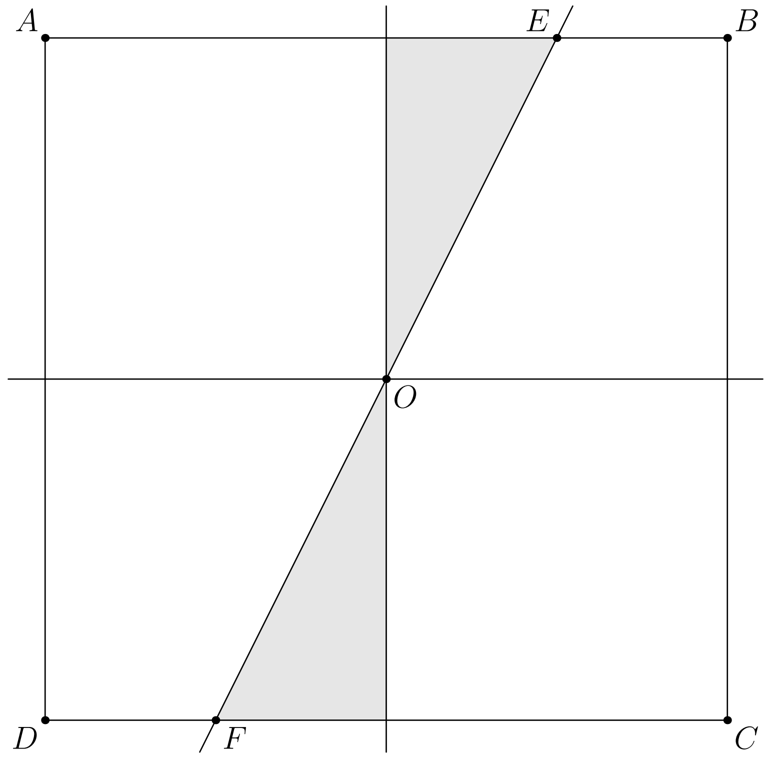
$$\begin{aligned}[\triangle AFD] + [\triangle ADE] - [\triangle BFC] &= \frac{8^2\sqrt{3}}{4} + \frac{8^2\sqrt{3}}{4} - \frac{4^2\sqrt{3}}{4} \\ &= 32\sqrt{3} - 4\sqrt{3} \\ &= \boxed{28\sqrt{3}}.\end{aligned}$$

**8.** **[6]** Akliu randomly selects a point on the $xy$-plane inside the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$. He then draws a line through that point and the origin. What is the probability his line has a slope greater than 2?

*Proposed by Kevin Shen.*

**Answer:** $\boxed{\frac{1}{8}}$

**Solution:**




Let the point Akliu chooses be $P(a,b)$. Then, the line through $P$ and the origin $O(0,0)$ will be $y = \frac{b}{a}x$.

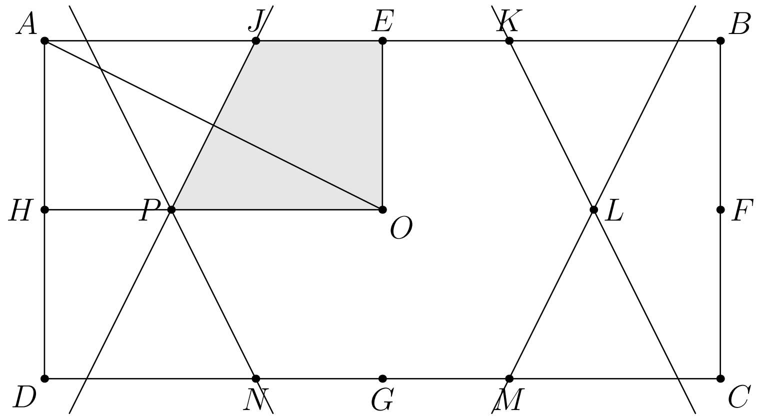
Now, consider the line $y = 2x$. The line will have slope $b/a$ greater than 2 if the point $P$ lies within the shaded region above, where $FE$ is the line $y = 2x$, by increasing the rise of the slope. Since $E$ is a distance $1/2$ from the $y$-axis, this area is $1/8$ of the square's area, so the answer is $\boxed{\frac{1}{8}}$.

**9.** **[7]** Daniel has a 4 inch by 8 inch piece of rectangular origami paper. He first chooses a vertex of the paper, then folds the vertex inward in one step so that the vertex is mapped to the center of the rectangle, and then unfolds the paper, leaving a straight crease in the paper. He then repeats this process for each of the remaining vertices. After this process, 4 creases are left in the paper. What is the area of the hexagon bounded by these 4 creases and the paper?

*Proposed by Sam Easaw.*

**Answer:** $\boxed{16}$

**Solution:**




Refer to the diagram above. Consider the fold at vertex $A$ to $O$. Then, the crease that is left will be the perpendicular bisector of $AO$. By symmetry, the perpendicular bisectors of $AO$ and $DO$ will intersect at $P$ on the midline $OH$ of $ABCD$.

So, we can split the hexagon $JKLMNP$'s area into areas covered by 4 smaller rectangles. We will focus on rectangle $AEOH$.

Note that the perpendicular bisector of $AO$ passes through the midpoint of $AO$. This means that it splits the area of $AEOH$ exactly in half. Thus, the perpendicular bisectors will all split the area of $ABCD$ exactly in half, so the answer is $\frac{1}{2} \cdot 32 = \boxed{16}$.

**10.** **[8]** For some positive integer $n$, let $f(n)$ output the smallest positive integer such that $n + f(n)$ has exactly 20 factors. What is $f(f(f(26)))$?

*Proposed by Lewis Lau.*

**Answer:** $\boxed{214}$

**Solution:** We can start by computing $f(26)$.

The possible ways for a number $N$ to have exactly 20 factors are for $N$ to have a prime factorization of one of the following forms:

– $p_1^{19}$

– $p_1^9 p_2$

– $p_1^4 p_2^3$

– $p_1^4 p_2 p_3$

For any of these, they are minimized for $p_1 < p_2 < p_3$, so the minimal values are $2^{19} = 524288$, $2^9 \cdot 3 = 1536$, $2^4 \cdot 3^3 = 432$, and $2^4 \cdot 3 \cdot 5 = 240$. The smallest is 240, so $f(26) = 240 - 26 = 214$.

Now, it's clear to see that $f(214)$ is just $240 - 214 = 26$, since we already computed the smallest $N$ that has 20 factors. Thus,

$$f(f(f(26))) = f(f(214)) = f(26) = \boxed{214}.$$

Note that, if $g(n)$ is the smallest integer greater than $n$ that has 20 factors, and $n \le g(n)$, then $f(f(n)) = n$. We know that $n + f(n) = g(n)$, and if $n$ and $f(n)$ are both less than $g(n)$, applying the function to $f(n)$ just outputs $n$ again, as they are symmetric about $g(n)$ and $f(n) < g(n)$ as well.

**11.** [**8**] A unit fraction is any simplified fraction with a numerator of 1. What is the sum of all unit fractions whose decimal representations terminate?

*Proposed by Evan Zhang.*

**Answer:** $\boxed{\frac{5}{2}}$

**Solution:** Notice that a fraction's decimal representation terminates if and only if its denominator is divisible only by 2 and 5. That means all terminating unit fractions can be written in the form $\frac{1}{2^{e_1} \cdot 5^{e_2}}$ for nonnegative integers $e_1$ and $e_2$. The sum of all terminating unit fractions can therefore be written as

$$\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots\right) = \frac{1}{1 - \frac{1}{2}} \times \frac{1}{1 - \frac{1}{5}} = 2 \times \frac{5}{4} = \boxed{\frac{5}{2}}.$$

**12.** [**9**] For some positive integer $n$, there exists a positive integer $d$ such that $n$ has $6d$ factors, $2n$ has $8d$ factors, and $3n$ has $9d$ factors. What is the sum of all possible integers $n$ for $n \le 150$?

*Proposed by Eric Xie.*

**Answer:** $\boxed{288}$

**Solution:** Let $d(n)$ be the number of factors of $n$.

Let the exponent of 2 in the prime factorization of $n$ be $a$, and the exponent of 3 be $b$. Thus 2 contributes a factor of $a+1$ to $d(n)$, and 3 contributes a factor of $b+1$ to $d(n)$.

The exponent of 2 in the prime factorization of $2n$ is $a+1$, so 2 contributes a factor of $(a+1)+1 = a+2$ to $d(2n)$. Thus we have

$$\frac{8d}{6d} = \frac{4}{3} = \frac{a+2}{a+1}$$

which means $a = 2$.

Similarly, if we analyze the exponent of 3, we get

$$\frac{9d}{6d} = \frac{3}{2} = \frac{b+2}{b+1}$$

thus $b = 1$.

Therefore $n = 2^2 \cdot 3 \cdot k$ for some positive integer $k$. Importantly, $k$ cannot contain any factors of 2 or 3 (as that would change $a$ and $b$). Notice that $12k \leq 150$ which means $k \leq 12$. Thus $k$ can be $1, 5, 7, 11$ so the answer is $12(1+5+7+11) = \boxed{288}$.

**13.** **[9]** Lewis is standing at the origin of the $xy$-plane. At any given moment, he has a $1/2$ chance of moving 1 unit upward and a $1/2$ chance of moving 1 unit to the right. Given that he never reaches the point $(3,3)$, what is the probability that Lewis will reach the point $(4,4)$?

*Proposed by Lewis Lau.*

**Answer:** $\boxed{\frac{15}{88}}$

**Solution:** Consider Lewis's location after 6 movements. With $\binom{6}{i}$ paths to $(i, 6-i)$, there is a $\binom{6}{i}/64$ chance Lewis is at $(i, 6-i)$. For Lewis to reach $(4,4)$, he must not be at any of $(6,0)$, $(5,1)$, $(1,5)$, or $(0,6)$. Furthermore, we know he can't reach $(3,3)$. Thus, he must be at either $(4,2)$ or $(2,4)$.

In the case of $(4,2)$, he must move up twice. Otherwise, he will be too far to the right to ever reach $(4,4)$. Thus, there is a $\binom{6}{2}/64/4 = 15/256$ chance to reach $(4,2)$ and $(4,4)$. By symmetry, the probability is the same for $(2,4)$. Since the chance of reaching $(3,3)$ is $\binom{6}{3}/64 = 20/64$, the final answer is

$$\frac{2 \cdot \frac{15}{256}}{1 - 20/64} = \boxed{\frac{15}{88}}.$$

**14.** **[10]** What is the remainder when

$$2026^{2025+2024^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}}}$$

is divided by 101?

*Proposed by Bill Qian.*

**Answer:** $\boxed{17}$

**Solution:** First, since 101 is prime, we have that by Fermat's Little Theorem,

$$2026^{101-1} = 2026^{100} \equiv 1 \pmod{101}$$

since $2026 = 1013 \cdot 2$ is relatively prime to 101. Thus, we can express the desired quantity as

$$2026^{2025+2024^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}}} = 2026^{100d+k} = \left(2026^{100}\right)^d \cdot 2026^k \equiv 2026^k \pmod{101}.$$

We wish to find $k$, which is the remainder when the exponent is divided by 100. Taking modulo 100,

$$2025 + 2024^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}} \equiv 25 + 24^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}} \pmod{100}.$$

To evaluate this, we may use the Chinese Remainder Theorem. We know that, taking modulo 4,

$$25 + 24^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}} \equiv 1 + 0 = 0 \pmod{4}$$

and that, taking modulo 25,

$$25 + 24^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}} \equiv 0 + (-1)^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}} \equiv -1 \equiv 24 \pmod{25}$$

since $2023 + 2022^{2021+2020^{\cdot^{\cdot^{\cdot^{3+2^1}}}}}$ is odd.

Since $100d + k$ is $0 \pmod{4}$ and $24 \pmod{25}$, the Chinese Remainder Theorem implies that it must be $49 \pmod{100}$. Thus, $k = 49$. We can now evaluate the remainder:

$$2026^{2025+2024^{2023+2022^{\cdot^{\cdot^{\cdot^{3+2^1}}}}}} \equiv 2026^{49} \equiv (2020+6)^{49} \equiv 6^{49} \pmod{101}.$$

Since $6^5 \equiv -1 \pmod{101}$, we have that

$$6^{49} \equiv 6^{45} \cdot 6^4 \equiv (-1)^{45} \cdot 1296 \equiv -1296 \equiv 1313 - 1296 \equiv \boxed{17} \pmod{101}.$$

**15.** [**10**] $\triangle ABC$ is triangle such that the foot of the altitude from $A$ to $\overline{BC}$ is $D$, and the line $AD$ intersects the circumcircle of $\triangle ABC$ at a point $E \neq A$. Let $M$ be the midpoint of $\overline{BC}$. Given that $BD = 6$, $DC = 7$, and that circumcircle of triangle $\triangle MDE$ is tangent to the circumcircle of $\triangle ABC$, what is the area of triangle $\triangle ABC$?

*Proposed by Tane Park.*

**Answer:** $\boxed{\dfrac{13\sqrt{42}}{2}}$

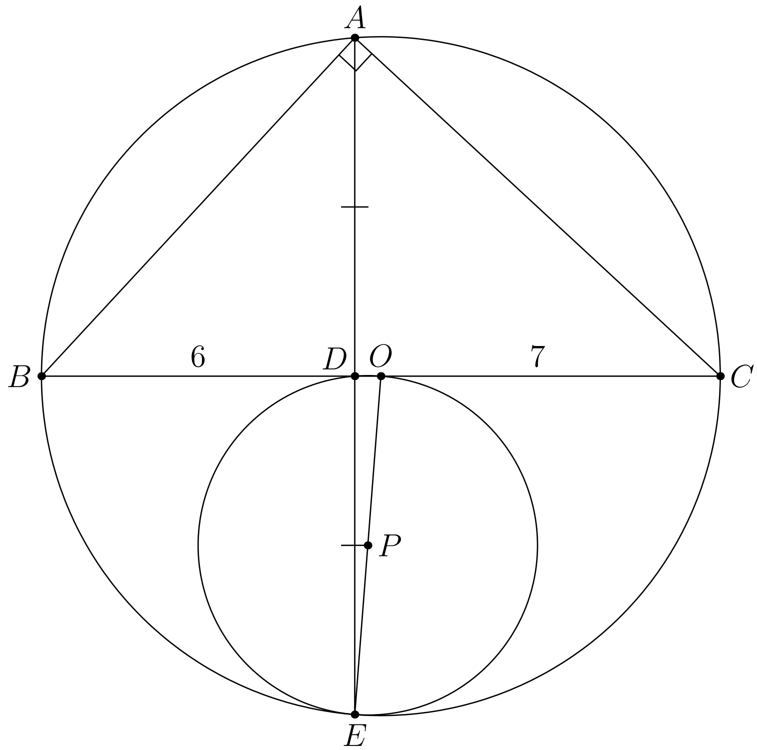
**Solution:** Let $(ABC)$ denote the circle passing through points $A$, $B$, and $C$.

Let $O$ be the center of $(ABC)$, and let $P$ be the center of $(MDE)$. Since $M$ lies inside of $(ABC)$, $(MDE)$ must be internally tangent to $(ABC)$ at $E$. This implies that points $O$, $P$, and $E$ lie on one line. Then, since $\angle MDE = 90°$, $P$ must be the midpoint of $\overline{ME}$, and thus $O$, $P$, $E$, and $M$ all lie on one line.

Since line $OM$ splits segment $\overline{BC}$ in half and intersects minor arc $BC$ at its midpoint, $F$, this must imply that $E = F$, and that $\triangle ABC$ is isosceles. However, since $BD \neq CD$, this is a contradiction. Now, since we have assumed nothing about the triangle except that $O$, $P$, $E$, and $M$ are distinct for this contradiction, they must thus not all be distinct.

Since $P$ is the midpoint of $\overline{EM}$, points $P$, $E$, and $M$, must be distinct unless $E = M$, which cannot occur since $E$ lies on $(ABC)$ and $M$ lies on line $BC$, which already intersects $(ABC)$ at $B$ and $C$.

Thus, $O$ must be any of $P$, $E$, and $M$. $O$ cannot be $E$, as it does not lie on $(ABC)$, and cannot be $P$, as that would imply $M$ is the reflection of $E$ across $O$ and thus lies on $(ABC)$, which cannot be the case as before. Thus, $O = M$, which occurs when $\angle BAC = 90°$.




Note that by power of a point on $D$,

$$AD \cdot DE = BD \cdot CD = AD^2 = 6 \cdot 7$$

so $AD = \sqrt{42}$, and the area of $\triangle ABC$ is

$$\frac{1}{2} \cdot AD \cdot BC = \boxed{\frac{13\sqrt{42}}{2}}.$$