1. What is the smallest positive perfect square that is divisible by 8?

Proposed by Sam Easaw.

Answer: $\boxed{16}$

Solution: For a perfect square to be divisible by 8, it needs to be the square of a multiple of 4. The smallest multiple of 4 is 4, and $4^2 = \boxed{16}$.

2. What is the sum of the smallest and largest 2-digit integers whose squares end in a 6?

Proposed by Evan Zhang.

Answer: $\boxed{110}$

Solution: The only way for a square k^2 to end in a 6 is if k has units digit 4 or 6. It is easy to check that the other possible units digits of k give k^2 a units digit that is not 6. Thus, the smallest 2-digit integer that ends in a 4 or 6 is 14, and the largest is 96, so the answer is $14 + 96 = \boxed{110}$.

3. Kenneth is thinking of a 3-digit number. When he removes the units digit of the number, his new number is divisible by 15. When he removes the hundreds digit of the number, his new number is still divisible by 15. What is the largest possible 3-digit number that Kenneth could be thinking of?

Proposed by Sam Easaw.

Answer: $\boxed{900}$

Solution: Let the number when he removes the units digit be U and the number when he removes the hundreds digit be H. Consider that U and H must have a units digit of either 0 or 5, since they are divisible by 15. If U has units digit 5, then H can be either 50 or 55, but since neither of these are divisible by 15, U must have units digit of 0. Thus H can either be 00 or 05, so H is just 00. We can now pick a hundreds digit that maximizes U. Since 90 is the largest 2-digit multiple of 15 that ends in 0, the answer is $\boxed{900}$.

4. A semiprime is a number that is the product of two distinct primes. Yunyi takes the product of two semiprimes and finds the resulting number has n factors. What is the sum of all possible values of n?

Proposed by Evan Zhang.

Answer: $\boxed{37}$

Solution: Let one semiprime be $m = p \cdot q$ and the other be $n = r \cdot s$, for some primes $p \neq q$ and $r \neq s$. Thus, $mn = pqrs$. We can split this product into three cases:

- If $p \neq q \neq r \neq s$, then $mn = pqrs$ has $n = 2^4 = 16$ factors.
- If $p = r, q \neq s$, then $mn = p^2qs$ has $n = 3 \cdot 2^2 = 12$ factors.
- If $p = r, q = s$, then $mn = p^2q^2$ has $n = 3^2 = 9$ factors.

Thus, the answer is $16 + 12 + 9 = \boxed{37}$.

5. How many positive integer divisors of 20^{26} have an odd number of divisors?

Proposed by Siyuan Yan.

Answer: $\boxed{378}$

Solution: Note that $20^{26} = (4 \cdot 5)^{26} = 2^{52} \cdot 5^{26}$. All factors of this number can be expressed as $d = 2^m 5^n$, for some $0 \leq m \leq 52$ and $0 \leq n \leq 26$. The number of divisors of d is given by $(m + 1)(n + 1)$. For this number to be odd, m and n must both be even, as $m + 1$ and $n + 1$ will both be odd. Thus, we can choose m from $0, 2, \ldots, 52$, and n from $0, 2, \ldots, 26$. We thus have 27 possible values of m and 14 possible values of n, giving an answer of $27 \cdot 14 = \boxed{378}$.

6. Let $S(n)$ denote the sum of digits of n. How many positive integers $k < 1000$ exist such that there exists some positive integer n that satisfies $k = n - S(n)$?

Proposed by Tane Park.

Answer: $\boxed{99}$

Solution: Each time n increases by 1, $S(n)$ either increases by 1 as well by increasing the units digit, or decreases by carrying the units digit when the units digit is 9. Now, consider $n - S(n)$. Note that when the units digit of n is not 9, we see that $(n + 1) - S(n + 1) = n + 1 - (S(n) + 1) = n - S(n)$. When the units of n is 9, digit carry happens, so $S(n)$ decreases whereas n increases and thus the value of $n - S(n)$ must always strictly increase. Thus, we can say that $n - S(n)$ is a non-decreasing quantity. Now, we can note that the sets

$$\{1, ..., 9\}, \{10, ..., 19\}, ..., \{990, \ldots, 999\}, \{1000, \ldots, 1009\}$$

each produce a unique value less than or equal to 1000:

$$0, 9, \ldots, 972, 999$$

as each successive set represents one digit carry, increasing the value of $n - S(n)$. Here, every set except for the first set produces a positive value, and every set after the last set will produce a value greater than 1000. Therefore, our answer is $(1000/10) - 1 = \boxed{99}$.

7. Let $f(n) = \text{lcm}(1, 2, 3, \ldots, n)$. For how many positive integers $n < 100$ does $f(n) = f(n+1)$?

Proposed by Thomas Yao.

Answer: $\boxed{64}$

Solution: We are trying to find all integers $n < 100$ that satisfy

$$\text{lcm}(1, 2, 3, \ldots, n) = \text{lcm}(1, 2, 3, \ldots, n, n+1).$$

The only difference between these two terms is the addition of an $n+1$ to the second term. In order for the addition of $n+1$ to not affect $f(n)$, $n+1$ must contain no prime factors that are not factors of $1, 2, \ldots, n$, as that would increase the lcm. Thus, if the addition of $n+1$ were to affect $f(n)$, $n+1$ must contain a new prime factor that doesn't appear in $1, 2, \ldots, n$. This occurs only when $n+1$ is a power of a prime, as p^k only shares factors with $a \cdot p^k$ for some integer a, but $a \cdot p^k$ is greater than p^k and thus does not appear in the divisors of $1, 2, \ldots, p^k - 1$. We can now compute the complement.

The number of primes less than 100 is known to be 25. There are 9 squares less than 100, and 4 of them are prime powers (2^2, 3^2, 5^2, 7^2). Similarly, there are 4 cubes less than 100, and 2 of them are prime powers (2^3, 3^3). There are 3 fourth powers less than 100, and 2 of them are prime powers (2^4, 3^4). Now, we are left with powers of 5 and 6, giving only 2^5 and 2^6 as prime powers. Thus, there are $25 + 4 + 2 + 2 + 1 + 1 = 35$ prime powers less than 100, so the answer is $99 - 35 = \boxed{64}$.

8. Define the function $f(n)$ as

$$f(n) \coloneqq \sum_{i=1}^{n} \gcd(i, n).$$

Sam takes the product of the first 8 primes and gets a number k. How many factors does $f(k)$ have?

Proposed by Evan Zhang.

Answer: $\boxed{384}$

Solution: We can group each of the i by the value of $\gcd(i, n)$. Since $\gcd(i, n) \mid n$, so pick some divisor $d \mid n$ to compute the sum. We would like to count all the i such that $\gcd(i, n) = d$.

Any such i would have $d \mid i$, so $i = a \cdot d$. Similarly, $n = (n/d) \cdot d$. Then, if $\gcd(i, n) = d$, we require that $\gcd(a, n/d) = 1$. With $1 \leq a \leq n/d$ (in order for $1 \leq i \leq n$), we have that the number of valid a is $\varphi(n/d)$ where φ is Euler's totient function. Then,

$$f(n) = \sum_{d|n} d \cdot \varphi(n/d).$$

Consider the prime factorization $k = 2^1 \cdot 3^1 \cdot \cdots \cdot 19^1$, since 19 is the 8th smallest prime. Any divisor $d \mid n$ can be written as $d = 2^{d_1} \cdot \cdots \cdot 19^{d_8}$ for all $d_j \in \{0, 1\}$. We can rewrite the sum as

$$\begin{aligned} f(k) &= \sum_{d_1, d_2 \ldots, d_8 \in \{0,1\}} d \cdot \varphi\left(\frac{k}{d}\right) \\ &= \sum_{d_1, d_2 \ldots, d_8 \in \{0,1\}} 2^{d_1} \times \cdots \times 19^{d_8} \times \varphi\left(\frac{2 \times \cdots \times 19}{2^{d_1} \times \cdots \times 19^{d_8}}\right) \\ &= \sum_{d_1, d_2 \ldots, d_8 \in \{0,1\}} \left(2^{d_1} \times \cdots \times 19^{d_8}\right)\left(\varphi\left(2^{1-d_1} \times \cdots \times 19^{1-d_m}\right)\right) \end{aligned}$$

Because φ is multiplicative, we can write this as

$$\begin{aligned} &\sum_{d_1=0}^{1} \sum_{d_2=0}^{1} \cdots \sum_{d_8=0}^{1} \left(2^{d_1} \times \cdots \times 19^{d_8}\right)\left(\varphi\left(2^{1-d_1} \times \cdots \times 19^{1-d_m}\right)\right) \\ &= \left(\sum_{d_1=0}^{1} 2^{d_1} \varphi\left(2^{1-d_1}\right)\right) \cdots \left(\sum_{d_8=0}^{1} 19^{d_8} \varphi\left(19^{1-d_8}\right)\right) \\ &= \left(2^0 \cdot \varphi(2^1) + 2^1 \cdot \varphi(2^0)\right) \cdots \left(19^0 \cdot \varphi(19^1) + 19^1 \cdot \varphi(19^0)\right) \\ &= (2 - 1 + 2) \cdots (19 - 1 + 19) \\ &= (2 \cdot 2 - 1) \cdots (2 \cdot 19 - 1) \end{aligned}$$

since $\varphi(1) = 1$ and $\varphi(p) = p - 1$ for some prime p.

So, our answer of $f(k)$ is just the product of $2p - 1$ for the first 8 primes p. Thus,

$$\begin{aligned} f(k) &= 3 \cdot 5 \cdot 9 \cdot 13 \cdot 21 \cdot 25 \cdot 33 \cdot 37 \\ &= 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \end{aligned}$$

which has $6 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = \boxed{384}$ factors.