1. **[3]** Sam, Bill, and David went to an all-you-can-eat sushi restaurant and ate 70 pieces of sushi in total. Sam says he ate a prime number of pieces of sushi. David says that he ate 25 pieces of sushi. Bill says that the number of pieces he ate is divisible by 13. How many pieces of sushi did Sam eat?

Proposed by Sam Easaw.

Answer: $\boxed{19}$

Solution: David ate 25 pieces, so Sam and Bill ate $70 - 25 = 45$ pieces in total.

- If Bill ate 0 pieces, then Sam ate 45 pieces, which is not prime.
- If Bill ate 13 pieces, then Sam ate 32 pieces, which is not prime.
- If Bill ate 26 pieces, then Sam ate 19 pieces, which is prime.
- If Bill ate 39 pieces, then Sam ate 6 pieces, which is not prime.

So, Sam must have eaten $\boxed{19}$ pieces.

2. **[3]** What is the sum of all two-digit numbers whose digits multiply to 20?

Proposed by Sam Easaw.

Answer: $\boxed{99}$

Solution: The only way two single-digit numbers can multiply to 20 is with $4 \cdot 5$ or $5 \cdot 4$. Thus, the only two-digit numbers whose digits multiply to 20 are 45 and 54, so the answer is $45 + 54 = \boxed{99}$.

3. **[3]** A square has the same area and perimeter. What is its length?

Proposed by Evan Zhang.

Answer: $\boxed{4}$

Solution: Let the side length of the square be s. Then, $s^2 = 4 \cdot s$, so $s^2 - 4s = s(s-4) = 0$. s cannot be 0, so $s = \boxed{4}$.

4. **[3]** A fair 6-sided die is rolled once onto a table so that one of the faces is facing the table. What is the probability that the sum of the numbers on each of the 5 visible faces is odd?

Proposed by Sam Easaw.

Answer: $\boxed{\frac{1}{2}}$

Solution: The sum of the numbers on every face is $1 + 2 + 3 + 4 + 5 + 6 = 21$. Let k be the number that is face down on the table, so that $21 - k$ is the sum of the numbers on the visible faces. If k is odd, then $21 - k$ is even, and if k is even, then $21 - k$ is odd. Thus, we are looking for the probability that a random roll k is even, which is just $\boxed{\frac{1}{2}}$.

5. **[3]** Two brothers, Yao and Yur, are comparing their ages. When they add their ages together, they get 15. When they multiply them together, they get a perfect square. What is the age of the younger brother?

Proposed by Sam Easaw.

Answer: $\boxed{3}$

Solution: Go through each of the possible values of Yao and Yur's numbers.

- $7 + 8 = 15$, but $7 \cdot 8 = 56$, not a square.
- $6 + 9 = 15$, but $6 \cdot 9 = 54$, not a square.
- $5 + 10 = 15$, but $5 \cdot 10 = 50$, not a square.
- $4 + 11 = 15$, but $4 \cdot 11 = 44$, not a square.
- $3 + 12 = 15$, and $3 \cdot 12 = 36$, which is a square.
- $2 + 13 = 15$, but $2 \cdot 13 = 26$, not a square.
- $1 + 14 = 15$, but $1 \cdot 14 = 14$, not a square.

So, Yao and Yur's ages must be 3 and 12, in some order. We want the younger one, so the answer is $\boxed{3}$.

6. **[4]** The surface area of a cube is multiplied by 4 to form another cube. By what factor is the volume of the cube multiplied?

Proposed by Evan Zhang.

Answer: $\boxed{8}$

Solution: Let the side length of the cube be s. The surface area is given by $6s^2$. If this quantity is multiplied by 4, then it can be expressed as $4 \cdot 6s^2 = 6(2s)^2$, the surface area of a cube of side length $2s$. The volume of this new cube is $(2s)^3 = 8s^3$, which is 8 times larger than the original cube of volume s^3, so the answer is $\boxed{8}$.

7. [**4**] A palindrome is a number that is read the same forwards and backwards. For example, 11, 121, and 1331 are all palindromes. What is the only 3 digit palindrome that is divisible by 45?

Proposed by Sam Easaw.

Answer: $\boxed{585}$

Solution: If the palindrome is a 3-digit number, then its hundreds digit must not be zero, and thus its units digit must also not be zero since they must be the same. Since the number is divisible by 5, its first and last digits must be 5.

Our number is now $\underline{5X5}$, for some digit X. Since the number is divisible by 9, we know that $5 + X + 5 = 10 + X$ must be divisible by 9 as well. The only digit X which satisfies this is 8, as $10 + 8 = 18 = 2 \cdot 9$. Thus, the answer is $\boxed{585}$.

8. [**4**] Louie the cat has 3 bowls of milk. The first bowl has twice as much milk as the second bowl, and the third bowl has 6 more ounces of milk than the second bowl. If Louie pours all the milk together into one bowl, he has 30 ounces of milk in total. How many ounces of milk were in the first bowl?

Proposed by Gloria Jin.

Answer: $\boxed{12}$

Solution: Let the number of ounces in the first bowl be c. Then, the second bowl has $\frac{c}{2}$ ounces of milk, and the third bowl has $\frac{c}{2} + 6$ ounces of milk. Adding the bowls together,

$$\begin{aligned} c + \left(\frac{c}{2}\right) + \left(\frac{c}{2} + 6\right) &= 30 \\ c + c + 6 &= 30 \\ 2c &= 24 \\ c &= 12. \end{aligned}$$

Thus, the answer is $\boxed{12}$.

9. **[4]** Jeremy's school cafeteria offers 3 categories on their menu: 2 choices for hot food, 3 choices for sides, and 2 choices for drinks. How many different orders can Jeremy get for lunch if he can only pick at most one item from each category (Jeremy may choose to eat nothing)?

Proposed by Peter Guo.

Answer: $\boxed{36}$

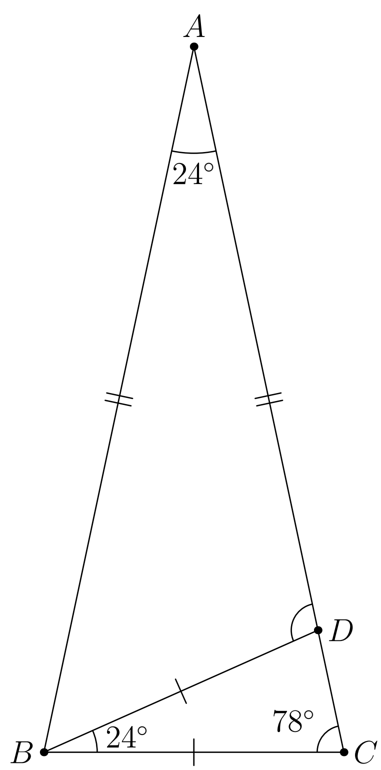
Solution: If Jeremy can choose at most 1 item from each category, he can not only choose the given options from the category but can also choose to eat nothing from a category. This adds 1 to each option in each category. Thus, the number of combinations of each lunch category is $(2+1)(3+1)(2+1) = 3 \cdot 4 \cdot 3 = \boxed{36}$.

10. **[4]** Isosceles triangle $\triangle ABC$ with $AB = AC$ has $\angle BAC = 24°$, and a point X is chosen on the interior of segment $\overline{AC}$ such that $BX = BC$. What is the measure of angle $\angle BXA$?

Proposed by Sam Easaw.

Answer: $\boxed{102°}$

Solution:



Since $AB = AC$, we also have that $\angle ABC = \angle ACB$, so $\angle ACB = \frac{1}{2}(180^\circ - 24^\circ) = 78^\circ$. Since $BX = BC$, we also have that $\angle XCB = \angle BXC = 78^\circ$, so $\angle BXA = 180^\circ - \angle BXC = 180^\circ - 78^\circ = \boxed{102^\circ}$.

11. [**5**] The four digit number $\underline{MBMT}$ is divisible by 10, 11, and 12. What is the value of $M + B + T$?

Proposed by Sam Easaw.

Answer: $\boxed{13}$

Solution: Since $\underline{MBMT}$ is divisible by 10, $T = 0$. Since $\underline{MBM0}$ is divisible by 11, $M + M \equiv B + 0 \pmod{11}$, so $2M = B \pmod{11}$.

Then, since $\underline{MBM0}$ is divisible by 12 and thus 3, $M + B + M \equiv 2M + B \equiv 0 \pmod 3$. Also, since $\underline{MBM0}$ is divisible by 12 and thus 4, $\underline{M0}$ must be divisible by 4 as they are the last two digits of $\underline{MBM0}$, so M must be divisible by 2. We can now check on all possible values of M.

- If $M = 0$, then $\underline{0B00}$ is not a 4-digit number.
- If $M = 2$, then B must equal $2M = 4$ for the number to be divisible by 11. However, $2M + B \equiv 4 + 4 \equiv 8 \neq 0 \pmod 3$, so $M \neq 2$.
- If $M = 4$, then B must equal $2M = 8$ for the number to be divisible by 11. However, $2M + B \equiv 8 + 8 \equiv 16 \neq 0 \pmod 3$, so $M \neq 4$.
- If $M = 6$, then B must equal $B \equiv 2M \equiv 12 \equiv 1 \mod 11$ for the number to be divisible by 11. However, $2M + B \equiv 12 + 1 \equiv 13 \neq 0 \pmod 3$, so $M \neq 6$.
- If $M = 8$, then B must equal $B \equiv 2M \equiv 16 \equiv 5 \mod 11$ for the number to be divisible by 11. Then, $2M + B \equiv 16 + 5 \equiv 21 \equiv 0 \pmod 3$, so $M = 8$.

Thus, our number is $\underline{MBMT} = \underline{8580}$, so the answer is $M+B+T = 8+5+0 = \boxed{13}$.

12. [**5**] A number is *cultured* if it is prime, and leaves a remainder of 2 when divided by 13. What is the sum of all two-digit *cultured* numbers?

Proposed by Jincheng (Kevin) Feng.

Answer: $\boxed{108}$

Solution: Note that all two-digit prime numbers must be odd. Thus, a *cultured* number cannot be 2 more than an even number, as then it would be even. We can now check all two-digit numbers which are two more than an odd multiple of 13:

- $13 \cdot 1 + 2 = 15$, which is not prime.
- $13 \cdot 3 + 2 = 41$, which is prime.
- $13 \cdot 5 + 2 = 67$, which is prime.
- $13 \cdot 7 + 2 = 93$, which is not prime.

Thus, the answer is $41 + 67 = \boxed{108}$.

13. **[5]** Winter has a pink marker, a blue marker, and a black marker. Each time she writes one letter of a word, she has to pick a different colored marker before she continues writing the word. In how many different ways can she color the word *"WHIPLASH"*?

Proposed by Sam Easaw.

Answer: $\boxed{384}$

Solution: She has 3 ways to pick the first color, then each proceeding color has to be different from the last color, so there are two options for each letter that isn't the first. So, the answer is $3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 3 \cdot 2^7 = \boxed{384}$.

14. **[5]** The first 2027 positive odd integers are arranged in a sequence in such a way that each number is the average of the two numbers adjacent to it in the sequence. What is the smallest possible value of the 1014th term of this sequence?

Proposed by William Roe.

Answer: $\boxed{2027}$

Solution: Let the ordering be $\{a_i\} = \{a_1, a_2, \ldots, a_{2027}\}$. We claim that the sequence must be either always increasing or always decreasing.

Since all a_i are distinct, we can assume without loss of generality that $a_1 < a_3$. Since a_2 is the average of a_1 and a_3, we can say that $a_1 < a_2 < a_3$. Now, since a_3 is the average of a_2 and a_4, we can say that

$$2a_3 = a_2 + a_4, \quad \text{so} \quad a_4 = 2a_3 - a_2.$$

Because $a_2 < a_3$, we can then say that $a_2 < a_3 < a_4$. Thus, $a_1 < a_2 < a_3 < a_4$. Applying the same logic to a_4, and then all a_i until a_{2027}, we find that

$$a_1 < a_2 < \cdots < a_{2027}.$$

If $a_1 > a_3$ instead, we can apply the same logic to achieve $a_1 > a_2 > \cdots > a_{2027}$. Thus, our sequence must be

$$1, 3, 5, \ldots, 4051, 4053$$

or its reverse.

The nth term of the increasing sequence is $2n - 1$. Thus, the 1014th term must be $2 \cdot 1014 - 1 = 2027$. If we reverse the sequence, the middle number does not change, so the answer is $\boxed{2027}$.

15. **[5]** Two externally tangent circles with radii 8 and 12 have centers X and Y respectively, and they intersect at a single point Z. A point O is chosen on the common tangent line to the two circles passing through Z so that $OY = OX + 1$. What is the value of $OX + OY$?

Proposed by Sam Easaw.

Answer: $\boxed{80}$

Solution: The tangent line through Z to both circles is perpendicular to XY, by tangency. So, by Pythagorean Theorem,

$$OX^2 - XZ^2 = OZ^2 = OY^2 - YZ^2.$$

Thus, $OX^2 - 64 = OY^2 - 144$, and $OY^2 - OX^2 = 144 - 64 = 80$. We can express this as

$$OY^2 - OX^2 = (OY + OX)(OY - OX)$$

and we are given that $OY - OX = 1$, so

$$OY^2 - OX^2 = (OX + OY) \cdot 1 = OX + OY = \boxed{80}.$$

16. **[7]** Kele is struggling on a true or false brain rot test. If the test has 7 questions in total, what is the probability that Kele randomly guesses at least 6 questions correct?

Proposed by Ashley Zhang.

Answer: $\boxed{\dfrac{1}{16}}$

Solution: For Kele to succeed, she must either guess 6 or 7 of the questions correctly.

If she guesses 6 correctly, there are $\binom{7}{1} = 7$ different ways she could have answered the questions (choosing which one she got wrong). Each problem has a 1/2 of being guessed correctly. Thus she has a $7/2^7 = 7/128$ chance of guessing 6.

If she guesses 7 correctly, there is only 1 way to do this. Each problem has a 1/2 chance of being guessed correctly. Thus there is a 1/128 chance that she guesses 7 correctly.

In total the probability that she guesses at least 6 correctly is

$$\frac{7}{128} + \frac{1}{128} = \boxed{\frac{1}{16}}.$$

17. **[7]** Let the Fibonacci sequence be defined by first term $F_1 = 1$, second term $F_2 = 1$, and nth term $F_n = F_{n-1} + F_{n-2}$. How many of the first 2026 terms of the Fibonacci sequence are odd?

Proposed by Siyuan Yan.

Answer: $\boxed{1351}$

Solution: Note that the parity (even or odd) of any F_n only depends on F_{n-2} and F_{n-1}. With finitely many such pairs, the parity must be periodic. Mapping out a few terms, with 1 denoting odd and 0 denoting even, we have

$$1, 1, 0, 1, 1, 0, \dots,$$

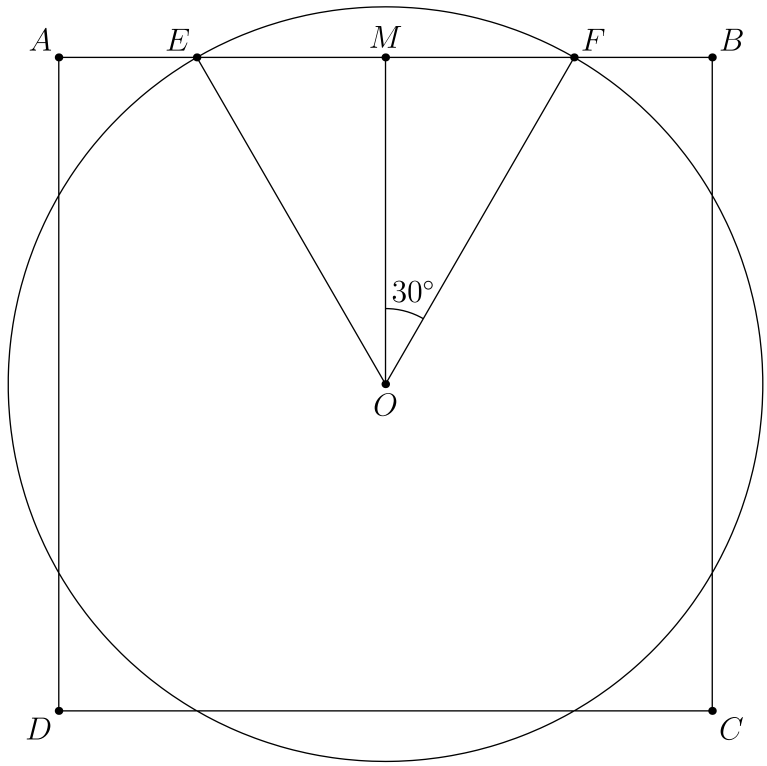
repeating every 3 terms. Thus, of the first 2025 terms, exactly $2 \cdot 2025/3 = 1350$ terms are odd. Furthermore, the 2026th term is odd, so there are $\boxed{1351}$ odd terms.

18. **[7]** A square and a circle share the same center, and the ratio of the square's side length to the circle's radius is $\sqrt{3}$. What fraction of the square's perimeter lies within the circle?

Proposed by Sam Easaw.

Answer: $\boxed{\frac{\sqrt{3}}{3}}$

Solution:



Let side length of the square be s and the radius of the circle be r. Then, $s/r = \sqrt{3}$, and thus $(s/2)/r = \sqrt{3}/2$.

Referring to the diagram above, we can thus say that $(s/2)/r = OM/OF = \sqrt{3}/2$. Thus, $\triangle OMF$ is a $30-60-90$ triangle. Since the figure is symmetric, we thus need to find EF/AB.

Assume $AB = 1$. Thus, $OM = 1/2$, and $MF = \frac{1}{2\sqrt{3}}$, and thus $EF = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Thus, the answer is $EF/AB = \boxed{\dfrac{\sqrt{3}}{3}}$.

19. **[7]** Let

$$f(x) = (x-1)(x-4)(x-9)(x-16)(x-25)\cdots(x-2025^2).$$

The value of $f(2026^2)$ can be expressed as $n!/k$ for some integers n and k, where k is minimized. What is the value of $n+k$?

Proposed by William Roe.

Answer: $\boxed{6077}$

Solution: Note that

$$\begin{aligned} f(2026^2) &= (2026^2-1)(2026^2-4)\cdots(2026^2-2025^2) \\ &= (2026+1)(2026-1)(2026+2)(2026-2)\cdots \\ &\qquad \cdots(2026+2025)(2026-2025) \\ &= 2027\cdot 2025\cdot 2028\cdot 2024\cdots 4051\cdot 1 \\ &= 1\cdot 2\cdots 2024\cdot 2025\cdot 2027\cdot 2028\cdots 4050\cdot 4051. \end{aligned}$$

This product is just 4051! with the 2026 term omitted. Thus, this quantity is equal to

$$f(2026^2) = \frac{1\cdot 2\cdot 3\cdots 4050\cdot 4051}{2026} = \frac{4051!}{2026}$$

which clearly minimizes k. Thus, the answer is $4051+2026=\boxed{6077}$.

20. **[7]** Thomas chooses three unique points at random from the vertices of a regular 2026-gon. What is the probability that the triangle formed by those three points is obtuse?

Proposed by Siyuan Yan.

Answer: $\boxed{\dfrac{337}{675}}$

Solution: Let ω be the circle passing through all the vertices of his polygon. Let one vertex of the polygon be A, and let the point diametrically opposite of A on ω and the polygon be B.

We can choose another point C from the rest of the points on the polygon. Now, we can choose another point D on the polygon to form $\triangle ACD$.

Note that $\angle ACB = 90^\circ$ since AB is a diameter of ω. Thus, if D were to lie on the other side of AB than C, then $\angle ACD < 90^\circ$ by the inscribed angle theorem, and clearly $\angle ADC < 90^\circ$ as well by the same argument. This would cause triangle $\triangle ACD$ to be acute.

Instead, we can choose C and D to be on the same side of AB, so that $\angle ACD$ or $\angle ADC$ is obtuse by the inscribed angle theorem, making triangle $\triangle ADC$ obtuse.

We may choose C and D from one side in

$$2\cdot\binom{(2026-2)/2}{2} = 2\cdot\binom{1012}{2}$$

ways, and we may choose C and D from the entire polygon in

$$\binom{2026-1}{2} = \binom{2025}{2}$$

ways.

Thus, the answer is

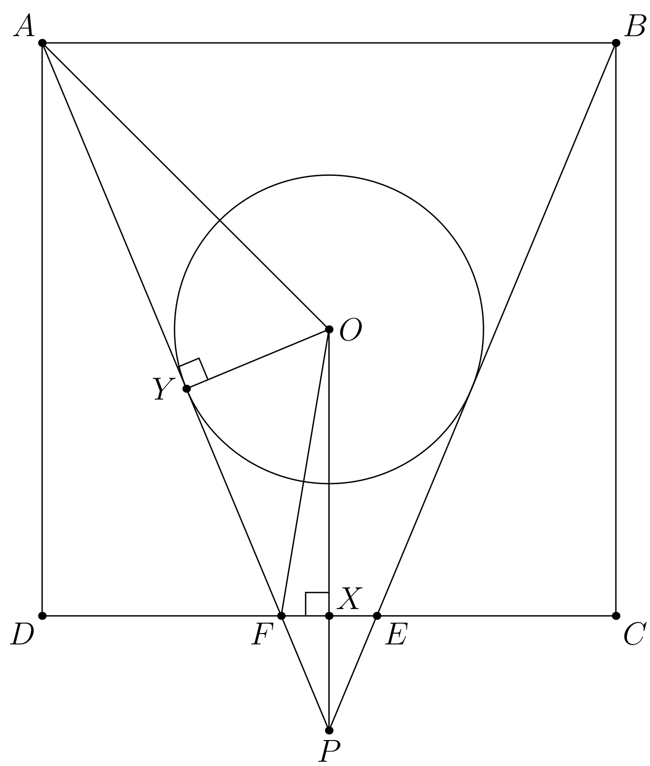
$$\begin{aligned}
\frac{2 \cdot \binom{1012}{2}}{\binom{2025}{2}} &= \frac{2 \cdot (1012 \cdot 1011/2)}{2025 \cdot (2024/2)} \\
&= \frac{1012 \cdot 1011}{2025 \cdot 1012} \\
&= \frac{1011}{2025} \\
&= \boxed{\frac{337}{675}}.
\end{aligned}$$

21. **[9]** Let $ABCD$ be a square with side length 24. Points E and F are chosen on segment $\overline{CD}$ such that $CE = DF = 10$. A circle tangent to segments $\overline{AF}$ and $\overline{BE}$ is centered at O, the center of the square. What is the radius of this circle?

Proposed by Eric Xie.

Answer: $\boxed{\frac{84}{13}}$

Solution:



Let X be the midpoint of CD and Y be the foot of the perpendicular line from O to AF. The radius of the circle is then OY.

Also, we have that $OX = 24/2 = 12$, $FX = DX - DF = 12 - 10 = 2$, and also $AF = \sqrt{DF^2 + AD^2} = \sqrt{10^2 + 24^2} = 26$.

Solution 1:

Let AF and BE intersect at a point P.

Note that ΔAFD is similar to ΔPFX as they are both right and $\angle AFD = \angle PFX$. Thus,

$$\begin{aligned} \frac{PX}{FX} &= \frac{AD}{FD} \\ PX &= \frac{24}{10} \cdot 2 \\ &= \frac{24}{5} \end{aligned}$$

Now, note that $OP = PX + OX = 24/5 + 12 = 84/5$. Then, $\triangle POY$ is similar to $\triangle PFX$, as they are both right and share an angle, and thus $\triangle POY$ is similar to $\triangle AFD$. So,

$$\begin{aligned}\frac{OY}{OP} &= \frac{FD}{AF}\\ OY &= \frac{10}{26}\cdot\frac{84}{5}\\ &= \boxed{\frac{84}{13}}\end{aligned}$$

Alternate Solution:

Note that OA is half the diagonal of $ABCD$, so OA^2 is equal to

$$OA^2 = \frac{1}{4}(24^2+24^2) = 12^2+12^2 = 288.$$

We can proceed by Pythagorean theorem. Note that

$$OF^2 = FX^2 + OX^2 = 2^2 + 12^2 = 148.$$

Then, letting $OY = r$, we have that

$$\begin{aligned}AY + YF &= \sqrt{OA^2 - r^2} + \sqrt{OF^2 - r^2}\\ &= \sqrt{288 - r^2} + \sqrt{148 - r^2}\\ &= AF = 26.\end{aligned}$$

Furthermore, we can say that

$$\left(\sqrt{288-r^2}\right)^2 - \left(\sqrt{148-r^2}\right)^2 = 140,$$

so

$$\sqrt{288-r^2} - \sqrt{148-r^2} = \frac{140}{26} = \frac{70}{13}.$$

Then,

$$\sqrt{288-r^2} = \frac{26+\frac{70}{13}}{2} = 204/13$$

and $r = \sqrt{288-(204/13)^2} = \boxed{84/13}$.

22. **[9]** Let $f(n)$ output the sum of the first n positive integers, where n is positive. Let S be the set of all integers k such that $f(k)$ is a perfect square. What is the sum of the 3 smallest numbers in S?

Proposed by Sam Easaw.

Answer: $\boxed{58}$

Solution: Note that $f(n) = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. By the Euclidean Algorithm, k and $k+1$ satisfy

$$\gcd(k, k+1) = \gcd(k, 1) = 1.$$

For $f(k)$ to be a perfect square, $k+1$ and $k/2$ themselves must be squares or alternatively $(k+1)/2$ and k must be squares.

We check upwards for two consecutive integers that satisfy $k = a^2$ and $k \pm 1 = 2 \cdot b^2$, for some a and b. We also only need to check odd square k as $k \pm 1$ is divisible by two only in this case.

- At $k = 1$, $k + 1 = 2 = 2 \cdot (1)^2$, so $k = 1$ works.
- At $k = 9$, $k - 1 = 8 = 2 \cdot (2)^2$, so $k + 1 = 9$, $k = 8$ works.
- At $k = 25$, $k - 1 = 24 = 2 \cdot 12$, $k + 1 = 26 = 2 \cdot 13$, doesn't work.
- At $k = 49$, $k + 1 = 50 = 2 \cdot (5)^2$, so $k = 49$ works.

So, the answer is $1 + 8 + 49 = \boxed{58}$.

23. **[9]** Grogg places 3 lamps around a circle and turns exactly one of them on. After every second, the lamp directly clockwise to any lamp that is on will turn either on or off with equal probability. What is the expected number of seconds that pass until all of the lamps are on at once?

Proposed by Sam Easaw.

Answer: $\boxed{8}$

Solution: We use states. Let E_n be the expected number of moves it takes to turn on all of the lamps when n lamps are on. We wish to find E_1.

Note that in the E_1 state, where 1 lamp is on, there is a 1/2 chance to have no change (the next bulb stays off) and a 1/2 chance to attain the E_2 state from E_1.

Then, in the E_2 state, where 2 lamps are on, there is a 1/4 chance of re-entering the E_1 state by turning one lamp off, a 1/2 chance of remaining in the E_2 state by keeping all lamps the same, and a 1/4 chance of entering the E_3 state from E_2 by turning on the last lamp. Thus, we can set up the following system:

$$E_1 = \frac{1}{2}(E_1 + 1) + \frac{1}{2}(E_2 + 1)$$
$$E_2 = \frac{1}{2}(E_2 + 1) + \frac{1}{4}(E_1 + 1) + \frac{1}{4}(E_3 + 1).$$

Note that E_3 is 0 by definition. Simplifying these equations gives $\frac{1}{2}E_1 = \frac{1}{2}E_2 + 1$ and $\frac{1}{2}E_2 = \frac{1}{4}E_1 + 1$. Thus, $\frac{1}{2}E_1 - 1 = \frac{1}{4}E_1 + 1$, and so $\frac{1}{4}E_1 = 2$ meaning $E_1 = \boxed{8}$.

24. [9] Evaluate the sum

$$\sum_{a=-9}^{9}\sum_{b=-9}^{9} \max\left(|a-b|, |a+b|\right).$$

Proposed by Tane Park.

Solution: Note that $|a-b|$ and $|a+b|$ represent the absolute distance between a and b, and the absolute distance between a and $-b$, respectively, on a number line, for some real numbers a and b. The maximum value of the distance between a and b and the distance between a and $-b$ is achieved when both a and b or both a and $-b$ have opposite sign (opposite sides of number line), as then their absolute distance is just $|a|+|b|$, the sum of their absolute distances from zero. Given some a and b, we can thus say that

$$\max(|a-b|, |a+b|) = |a| + |b|.$$

Now we can calculate

$$\begin{aligned}
\sum_{a=-9}^{9}\sum_{b=-9}^{9}(|a|+|b|) &= \sum_{a=-9}^{9}\sum_{b=-9}^{9}|a| + \sum_{a=-9}^{9}\sum_{b=-9}^{9}|b| \\
&= 2\cdot 19\cdot \sum_{a=-9}^{9}|a| \\
&= 2\cdot 19\cdot 2\cdot \sum_{a=1}^{9} a \\
&= 2\cdot 19\cdot 2\cdot 45 \\
&= \boxed{3420}.
\end{aligned}$$

25. [9] Square $ABCD$ has a side length of 12. A point P is drawn such that $AP = 6$ and $\angle BPD = 135°$. Let the line containing all such points P be ℓ, and let X be the point on ℓ such that the length of $\overline{AX}$ is minimized. What is the length of segment $\overline{CX}$?

Proposed by Sam Easaw.

Answer: $\boxed{\dfrac{33\sqrt{2}}{4}}$

Solution: Note that $AP = 6$ implies that P is a distance of 6 away from A, so P lies on the circle with radius 6 centered at A. Let this circle be ω.

Now, for the other condition, note that $\triangle BCD$ is a right triangle with $\angle BCD = 90^\circ$. Consider the circle with radius $BC = CD$ centered at C. By the inscribed angle theorem, any point Y outside the square and on the circle will satisfy $\angle BYD = \frac{1}{2}\angle BCD = 45^\circ$, and accordingly any point Z inside the square will satisfy $\angle BZD = 180^\circ - 45^\circ = 135^\circ$. We can thus conclude that, since $\angle BPD = 135^\circ$, P must lie on the circle with radius $BC = CD = 12$ centered at C. Let this circle be Ω. (P could lie on the circle at A with radius AB, but then $AP = 6$ could not hold, as $AB = 12$).

Thus, P must lie on both of these two circles. Circles ω and Ω are not tangent and must intersect as $AC = 12\sqrt{2}$ and $CP + AP = 12 + 6 > 12\sqrt{2} = AC$. Let the two points of intersection of ω and Ω be P and Q.

Both circles are symmetric about AC, and thus P and Q are symmetric about AC. Then, Q must be the reflection of P across AC, PQ is perpendicular to AC, and we are finding the distance from A to $\overline{PQ}$. We know that $\triangle APQ$ is isosceles, as $AP = AQ = 6$. Similarly, $\triangle CPQ$ is isosceles. Then, X is the foot of the altitude from A to PQ. Since AC is perpendicular to PQ, A, X, and C are collinear.

Now, by the Pythagorean Theorem,

$$CX^2 + XQ^2 = CQ^2 \quad \text{and} \quad AX^2 + XQ^2 = AQ^2.$$

Thus,

$$XQ^2 = CQ^2 - CX^2 = AQ^2 - AX^2$$

and

$$CX^2 - AX^2 = CQ^2 - AQ^2 = 12^2 - 6^2 = 108.$$

Since X lies on $\overline{AC}$, we have

$$CX + AX = 12\sqrt{2}.$$

Note that

$$CX^2 - AX^2 = (CX - AX)(CX + AX) = 108,$$

so

$$CX - AX = \frac{108}{CX + AX} = \frac{108}{12\sqrt{2}} = \frac{9}{\sqrt{2}} = \frac{9\sqrt{2}}{2}.$$

Now, we have a system of two variables in terms of CX and AX. Solving,

$$2 \cdot CX = (CX - AX) + (CX + AX) = \frac{9\sqrt{2}}{2} + 12\sqrt{2} = \frac{33\sqrt{2}}{2}$$

and finally

$$CX = \boxed{\frac{33\sqrt{2}}{4}}.$$

26. **[12]** Six points A, B, C, D, E, and F are chosen at random along the circumference of a unit circle. Estimate the expected value of the area of the intersection of triangles $\triangle ABC$ and $\triangle DEF$.

Proposed by Evan Zhang.

Answer: $\boxed{0.0864820984219}$

Solution: Note that the two triangles are independent, meaning the probability a given point inside the circle lies inside both triangle is equal to the probability it lies inside one triangle squared.

As a function of the distance from the center of the circle r, the probability a point is contained within a randomly chosen triangle is given by

$$f(r) = \frac{1}{4} - \frac{3}{2\pi^2}\operatorname{Li}_2(r^2)$$

where Li_2 represents the dilogarithm function.

We wish to evaluate

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \left(\frac{1}{4} - \frac{3}{2\pi^2}\operatorname{Li}_2(r^2)\right)^2 r\mathrm{d}r\mathrm{d}\theta \\
=&\pi\int_0^1 \frac{1}{16} - \frac{3}{4\pi^2}\operatorname{Li}_2(u) + \frac{9}{4\pi^4}\operatorname{Li}_2(u)^2\mathrm{d}u \\
=&\pi\left(\frac{1}{16} - \frac{3}{4\pi^2}\left(\zeta(2)-1\right) + \frac{9}{4\pi^4}\left(\frac{5}{2}\zeta(4) - 4\zeta(3) - 2\zeta(2) + 6\right)\right) \\
\approx&\boxed{0.0864820984219}
\end{aligned}$$

where $\zeta(s)$ denotes the Riemann zeta function.

27. **[12]** Let $!n! = n! \times (n-1)! \times \cdots \times 1!$ (where $m! = m \times (m-1) \times \cdots \times 1$). Estimate the number of trailing zeroes at the end of $!2026!$.

Proposed by Evan Zhang.

Answer: $\boxed{508785}$

Solution: A number has n trailing zeroes when it is divisible by $10^n = 2^n \times 5^n$. Factors of 2 are more common than factors of 5, so we wish to find the number of factors of 5 in !2026!, which is just the sum of the number of factors of 5 in $2026!, 2025!, \cdots, 2!, 1!$.

Every multiple of 5 contributes 1 factor of 5, every multiple of 25 contributes an additional factor, as does eveyr multiple of 125 and 625. Thus, we are trying to find

$$\sum_{n=1}^{2026}\left(\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n}{25}\right\rfloor+\left\lfloor\frac{n}{125}\right\rfloor+\left\lfloor\frac{n}{625}\right\rfloor\right).$$

(Note that all powers of 5 greater than 625 will be greater than 2026, so the floors are 0).

For convenience, we can make the sums start at $n = 0$. We can then evaluate the sum of each term independently as follows

$$\begin{aligned}
\sum_{n=0}^{2026}\left\lfloor\frac{n}{5}\right\rfloor &= 0\cdot 5+1\cdot 5+\cdots+404\cdot 5+405\cdot 2\\
&= \frac{(404)(405)}{2}\cdot 5+405\cdot 2 = 409860\\
\sum_{n=0}^{2026}\left\lfloor\frac{n}{25}\right\rfloor &= 0\cdot 25+1\cdot 25+\cdots+80\cdot 25+81\cdot 2\\
&= \frac{(80)(81)}{2}\cdot 25+81\cdot 2 = 81162\\
\sum_{n=0}^{2026}\left\lfloor\frac{n}{125}\right\rfloor &= 0\cdot 125+1\cdot 125+\cdots+15\cdot 125+16\cdot 27\\
&= \frac{(15)(16)}{2}\cdot 125+16\cdot 27 = 15432\\
\sum_{n=0}^{2026}\left\lfloor\frac{n}{625}\right\rfloor &= 0\cdot 625+1\cdot 625+2\cdot 625+3\cdot 152\\
&= \frac{(2)(3)}{2}\cdot 625+3\cdot 152 = 2331.
\end{aligned}$$

Summing all these terms gives

$$409860+81162+15432+2331 = \boxed{508785}.$$

To obtain a good estimate quickly, we can drop the floor functions and assume the pattern continues to infinity, giving

$$\sum_{n=0}^{2026}\frac{n}{5}+\frac{n}{25}+\cdots = \sum_{n=0}^{2026}\frac{n}{4} = \frac{1}{4}\frac{(2026)(2027)}{2} = 513337.75.$$

We know our answer will be less than this, and to see just how much more we can restrict the bounds of our infinite sum to only go up to $\frac{n}{625}$, giving a new estimate of 512516.41.

Clearly, our actual answer will still be less than this. Notice that $\lfloor\frac{n}{5}\rfloor$ will be less than $\frac{n}{5}$ approximately $\frac{4}{5}$ by approximately $\frac{1}{2}$. The same goes for the other terms, giving us a final estimate of

$$512516.41 - \frac{1}{2}\left(\frac{4}{5} + \frac{24}{25} + \frac{124}{125} + \frac{624}{625}\right) \cdot 2026 \approx 508717.$$

28. **[12]** A weighted 20 sided die has the numbers 1 through 20 written on its faces. The probability of rolling any face is proportional to the number on the face. Estimate the expected number of rolls it will take to roll each number 1 through 20 at least once.

Proposed by Evan Zhang.

Answer: $\boxed{263.584666762}$

Solution:

Similar to the case where each face has an equal probability to be rolled, we can approach this by summing the expected number of rolls to roll a new face. However, unlike the simpler case, this depends not only on how many unique faces have been rolled, but also what those faces are.

For any proper subset $Y \subsetneq X$, let $\mathbb{E}[Y]$ denote the expected number of rolls to roll a face not in Y and let $\sum Y$ denote the sum of the elements of Y. Then, the expected number of rolls $\mathbb{E}$ until all faces have been rolled is

$$\sum_{Y \subsetneq X} P(Y) \cdot \mathbb{E}[Y]$$

where $P(Y)$ is the probability that, at some point, Y is the set of unique faces rolled. At such a state, the probability we roll another value in Y is

$$\frac{1}{N} \sum_{y \in Y} y.$$

Then, the expected number of rolls for that to occur is

$$\mathbb{E}[Y] = N \cdot \left(\sum_{y \in Y} y\right)^{-1}.$$

We can compute $P(Y)$ recursively as follows:

$$P(Y) = \sum_{y \in Y} \frac{y}{N - \sum Y} \cdot P(Y \backslash \{y\})$$

where $Y \backslash \{y\}$ denotes the set of elements of Y apart from y. Since $Y \backslash \{y\} \subsetneq X$, we must have computed it anyways, so this isn't particularly computationally intensive.

Performing the necessary summation, we arrive at the value of $\boxed{263.584666762}$.

Note that the expected value of rolls to obtain every value at least once must be greater than the expected value of rolls to obtain a 1, which is 210.

29. **[12]** A real number x is chosen at random on the closed interval $[0, 1]$. Estimate the expected value of

$$\sum_{n=1}^{2026} \frac{\{nx\}}{\lceil nx \rceil}$$

where $\{x\}$ denotes the fractional part of x and $\lceil x \rceil$ denotes the least integer greater than x.

Proposed by Evan Zhang.

Answer: $\boxed{17.1853817526}$

Solution: Start by considering only the case where $n = k$, where we wish to find the expected value of

$$\frac{\{kx\}}{\lceil kx \rceil}.$$

Notice that for x satisfying $\frac{j-1}{k} < x \leq \frac{j}{k}$, for integer j, the fraction's numerator has an average of $\frac{1}{2}$ while the denominator is j. This means the total expected value when $n = k$ is

$$\sum_{j=1}^{k} \frac{1}{2jk}$$

giving a total expected value over all k as

$$\sum_{k=1}^{2026} \sum_{j=1}^{k} \frac{1}{2jk}.$$

We can see this value is equal to

$$\frac{\frac{1}{2}\left(\left(1 + \frac{1}{2} + \cdots + \frac{1}{2026}\right)^2 + \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{2026^2}\right)\right)}{2}.$$

This can be approximated as

$$\frac{1}{4}\left((\ln 2026+\gamma)^2+\frac{\pi^2}{6}\right)\approx \boxed{17.18}$$

where γ is the Euler-Mascheroni constant.

30. **[12]** For some integer x, let $k(x)$ denote the number of positive divisors of x multiplied by the sum of the positive divisors of x. For example, $k(12)=6\cdot 28=168$, as it has 6 divisors and they sum to 28. Let $\mathbb{N}$ denote the set of all natural numbers. Estimate

$$\sum_{x\in\mathbb{N}}\frac{k(x)}{x^3}.$$

Proposed by Bill Qian.

Answer: $\boxed{3.77047181129455556641}$

Solution: Let $f(x)=k(x)/x^3=s(x)\cdot d(x)/x^3$ where s is the sum of divisors and d is the number of divisors of x. Note that $f(x)$ is multiplicative since $s(x)$ and $d(x)$ are also multiplicative under relatively prime arguments. In particular, for relatively prime x and y, $f(xy)=f(x)f(y)$. Then, if the prime factorization of x is

$$p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k},$$

we can write

$$f(x)=f(p_1^{e_1})\cdot f(p_2^{e_2})\cdots f(p_k^{e_k}).$$

Then,

$$\sum_{x\in\mathbb{N}}f(x)=\prod_{i=1}^{\infty}\sum_{j=0}^{\infty}f(p_i^j)$$

where p_i is the ith smallest prime number.

We are thus interested in computing, for an arbitrary prime p,

$$\begin{aligned}
\sum_{k=0}^{\infty} f(p^k) &= \sum_{k=0}^{\infty} \frac{(k+1)(1+p+\cdots+p^k)}{p^{3k}} \\
&= \sum_{k=0}^{\infty} \frac{(k+1)(p^{k+1}-1)}{(p-1)p^{3k}} \\
&= \frac{1}{p-1} \sum_{k=0}^{\infty} \left(\frac{pk}{p^{2k}} + \frac{p}{p^{2k}} - \frac{k}{p^{3k}} - \frac{1}{p^{3k}} \right) \\
&= \frac{1}{p-1} \left(\frac{p \cdot p^{-2k}}{(1-p^{-2k})^2} + \frac{p}{1-p^{-2k}} - \frac{p^{-3k}}{(1-p^{-3k})^2} - \frac{1}{1-p^{-3k}} \right) \\
&= \frac{1}{p-1} \left(\frac{p}{(1-p^{-2})^2} - \frac{1}{(1-p^{-3})^2} \right) \\
&= \frac{1}{p-1} \left(\frac{p^5}{(p^2-1)^2} - \frac{p^6}{(p^3-1)^2} \right).
\end{aligned}$$

Taking the product over all primes gives $\boxed{3.7704718112945556641}$.