1. **[4]** Jaime has written 192 songs, with 24 to an album. How many albums does he have?

Proposed by William Roe.

Answer: $\boxed{8}$

Solution: If Jaime has 192 songs, with 24 songs in each album, and a albums, then $24 \cdot a = 192$. Then, he will have $a = 192/24 = \boxed{8}$ albums.

2. **[4]** What is the smallest prime number that leaves a remainder of 1 when divided by 50?

Proposed by Sam Easaw.

Answer: $\boxed{101}$

Solution: If the number leaves a remainder of 1 when divided by 50, it should be 1 more than a multiple of 50. The first number that satisfies this is 1, which is not prime. The second number is $50 + 1 = 51$, which is not prime. The third number is $100 + 1 = 101$, which is prime, so the answer is $\boxed{101}$.

3. **[4]** Spongebobert Squaretrousers lives in a cube under the sea of side length 2, and Spongerobert Rectanglepants lives in a rectangular prism above the sea with double the height, triple the length, and triple the width of Spongebobert's house. What is the volume of Spongerobert's house?

Proposed by Thomas Yao.

Answer: $\boxed{144}$

Solution: Spongebobert's cube initially has length 2, width 2, and height 2. After doubling the height, tripling the length, and tripling the width, the dimensions of Spongerobert's house should be length 6, width 6, and height 4. The volume of his rectangular prism is thus $6 \cdot 6 \cdot 4 = \boxed{144}$.

4. **[5]** Sam flips a fair coin 4 times. What is the probability that he flips the same face twice in a row, and then flips the other face twice in a row?

Proposed by Evan Zhang.

Answer: $\boxed{\dfrac{1}{8}}$

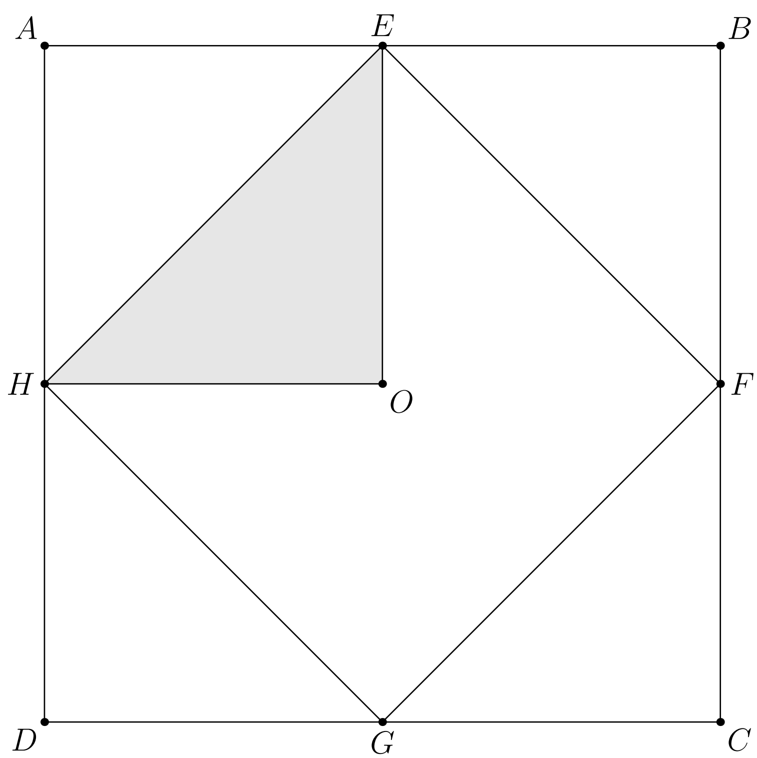
Solution: Evan can flip either two heads in a row and then two tails in a row, or two tails in a row and then two heads in a row, so only 2 combinations of flips. There are 2^4 possible combinations of 4 rolls he can make, and only 2 that work, so the answer is $2/16 = \boxed{\dfrac{1}{8}}$.

5. **[5]** Square $ABCD$ has side length 8. What is the area of the square formed by the midpoints of the sides of $ABCD$?

Proposed by Evan Zhang.

Answer: $\boxed{32}$

Solution:



Note that $ABCD$ can be split into four smaller squares, one of which is $AEOH$. Since $\triangle EOH$ is half the area of $AEOH$, we can say that $EFGH$ is half the area of $ABCD$. Thus, the answer is $\frac{1}{2} \cdot 8^2 = \boxed{32}$.

6. **[5]** Let $\lfloor x \rfloor$ denote the greatest integer less than x. What is the unique real number x that satisfies $x \cdot \lfloor x \rfloor = 10$?

Proposed by William Roe.

Answer: $\boxed{\dfrac{10}{3}}$

Solution: We know that

$$\lfloor x \rfloor^2 \leq x \cdot \lfloor x \rfloor \leq x^2.$$

Since, $x \cdot \lfloor x \rfloor = 10$, we have that $10 \leq x^2$, so $x \geq 3$. In the same way, $\lfloor x \rfloor^2 \leq 10$, so $\lfloor x \rfloor \leq 3$. Together, these conditions imply $\lfloor x \rfloor = 3$, so $3x = 10$ and thus $x = \boxed{\dfrac{10}{3}}$.

7. **[6]** Let a *half-prime* number be a positive integer n such that among the first n positive integers, *exactly* half of them are prime. What is the average of all half-prime numbers?

Proposed by Siyuan Yan.

Answer: $\boxed{5}$

Solution: We can begin by counting upwards with small n. For *exactly* half of the integers to be prime, n must be even.

- For $n = 2$, 2 is prime, so 2 is *half-prime*.
- For $n = 4$, 2 and 3 are prime, so 4 is *half-prime*.
- For $n = 6$, 2, 3, and 5 are prime, so 6 is *half-prime*.
- For $n = 8$, 2, 3, 5, and 7 are prime, so 8 is *half-prime*.
- For $n = 10$, 2, 3, 5, and 7, are prime, so 10 is not *half-prime*.

Now, assume all of the odd numbers less than n except 1 are prime. Thus, n would be half-prime. However, this is clearly not true for $n \geq 10$ as 9 is not prime. So, we are done. The average is thus $\frac{1}{4}(2 + 4 + 6 + 8) = \frac{1}{4}(20) = \boxed{5}$.

8. **[6]** Alice, Bill, and Chris each start with the same amount of apple juice in their cups. Alice pours 2/3 of her apple juice into Bill's cup, and afterwards Bill pours 4/5 of his current apple juice into Chris's cup. If they pour all of their juice into one bowl, what fraction of the juice in the bowl was poured from Chris's cup?

Proposed by Sam Easaw.

Answer: $\boxed{\dfrac{7}{9}}$

Solution: Assume Alice, Bill, and Chris, have 3 ounces of juice each. Then, Alice will give $3 \cdot 2/3 = 2$ ounces to Bill, leaving Alice with 1 ounce and Bill with $3 + 2 = 5$ ounces. Then, Bill gives $5 \cdot 4/5 = 4$ ounces to Chris, leaving Bill with 1 ounce and Chris with $3 + 4 = 7$ ounces. When they pour all of the juice into one container, 7 ounces out of the $3 + 3 + 3 = 9$ ounces total will come from Chris's cup. Thus, the answer is $\boxed{\dfrac{7}{9}}$.

9. **[7]** How many numbers between 1 and 1000 are divisible by 5 but not by 17?

Proposed by Lucas Zeng.

Answer: $\boxed{189}$

Solution: There are $1000/5 = 200$ numbers divisible by 5 between 1 and 1000. Note that all of these numbers can be written as $5 \cdot k$, where k ranges from 1 to 200.

So, we want k to not be divisible by 17. We can look for all k that are divisible by 17 and subtract that value from 200.

The smallest multiple of 17 less than 200 is $17 = 1 \cdot 17$, and the largest is is $187 = 11 \cdot 17$. So out of the 200 possible values of k, 11 of them are divisible by 17. Thus, the answer is $200 - 11 = \boxed{189}$.

10. **[8]** Let $ABCDE$ be a convex pentagon with segment $\overline{BC}$ parallel to segment $\overline{AD}$, $AB = BC = CD = 4$, and $DE = EA = AD = 8$. What is the area of pentagon $ABCDE$?

Proposed by Sam Easaw.

Answer: $\boxed{28\sqrt{3}}$

Solution: Since $DE = EA = AD = 8$, $\triangle ADE$ is an equilateral triangle of side length 8. Now, considering quadrilateral $ABCD$, since $AB = CD$ and $BC \parallel AD$, $ABCD$ must be an isosceles trapezoid of base lengths $AD = 8$ and $BC = 4$. If we extend lines AB and CD to meet at a point F, we see that since $AD/BC = 2$, $BF = CF = BC = 4$, by similar triangles. Thus, we can express the area of $ABCDE$ as

$$[\triangle AFD] + [\triangle ADE] - [\triangle BFC].$$

These are all equilateral triangles with side lengths 8, 8, and 4, respectively, so the desired area is

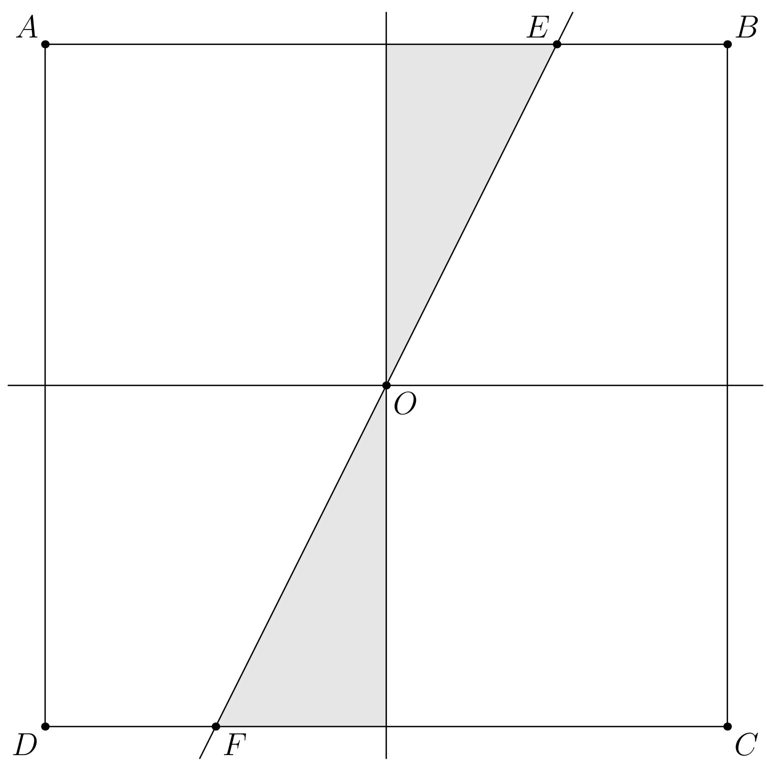
$$\begin{aligned}[\triangle AFD] + [\triangle ADE] - [\triangle BFC] &= \frac{8^2\sqrt{3}}{4} + \frac{8^2\sqrt{3}}{4} - \frac{4^2\sqrt{3}}{4} \\ &= 32\sqrt{3} - 4\sqrt{3} \\ &= \boxed{28\sqrt{3}}.\end{aligned}$$

11. [**8**] Akliu randomly selects a point on the xy-plane inside the square with vertices $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$. He then draws a line through that point and the origin. What is the probability his line has a slope greater than 2?

Proposed by Kevin Shen.

Answer: $\boxed{\frac{1}{8}}$

Solution:



Let the point Akliu chooses be $P(a,b)$. Then, the line through P and the origin $O(0,0)$ will be $y = \frac{b}{a}x$.

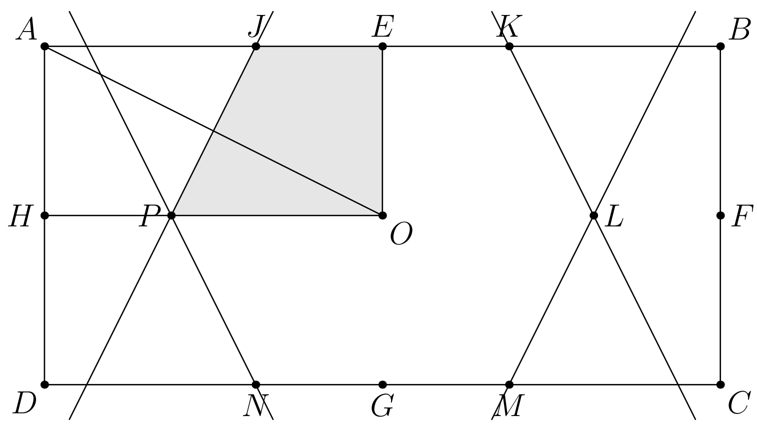
Now, consider the line $y = 2x$. The line will have slope b/a greater than 2 if the point P lies within the shaded region above, where FE is the line $y = 2x$, by increasing the rise of the slope. Since E is a distance $1/2$ from the y-axis, this area is $1/8$ of the square's area, so the answer is $\boxed{\frac{1}{8}}$.

12. **[9]** Daniel has a 4 inch by 8 inch piece of rectangular origami paper. He first chooses a vertex of the paper, then folds the vertex inward in one step so that the vertex is mapped to the center of the rectangle, and then unfolds the paper, leaving a straight crease in the paper. He then repeats this process for each of the remaining vertices. After this process, 4 creases are left in the paper. What is the area of the hexagon bounded by these 4 creases and the paper?

Proposed by Sam Easaw.

Answer: $\boxed{16}$

Solution:



Refer to the diagram above. Consider the fold at vertex A to O. Then, the crease that is left will be the perpendicular bisector of AO. By symmetry, the perpendicular bisectors of AO and DO will intersect at P on the midline OH of $ABCD$.

So, we can split the hexagon $JKLMNP$'s area into areas covered by 4 smaller rectangles. We will focus on rectangle $AEOH$.

Note that the perpendicular bisector of AO passes through the midpoint of AO. This means that it splits the area of $AEOH$ exactly in half. Thus, the perpendicular bisectors will all split the area of $ABCD$ exactly in half, so the answer is $\frac{1}{2} \cdot 32 = \boxed{16}$.

13. **[9]** Rectangle $ABCD$ has side lengths $AB = 10$ and $BC = 20$. Let E be the intersection of segments $\overline{AC}$ and $\overline{BD}$. There exists two distinct points F and G on line BC such that $BE = BF = BG$. What is the area of triangle $\triangle EFG$?

Proposed by Sam Easaw.

Answer: $\boxed{25\sqrt{5}}$

Solution: First, we know that $BF = \sqrt{5^2 + 10^2} = 5\sqrt{5}$. Thus, $EG = 2 \cdot (5\sqrt{5}) = 10\sqrt{5}$. The distance from point E to side BG is half of AB, so $BG = 10/2 = 5$, so the area of $\triangle EFG$ is just half of the base times height:

$$\frac{1}{2} \cdot 10 \cdot 5\sqrt{5} = \boxed{25\sqrt{5}}.$$

14. **[10]** For some positive integer n, there exists a positive integer d such that n has $6d$ factors, $2n$ has $8d$ factors, and $3n$ has $9d$ factors. What is the sum of all possible integers n for $n \leq 150$?

Proposed by Eric Xie.

Answer: $\boxed{288}$

Solution: Let $d(n)$ be the number of factors of n.

Let the exponent of 2 in the prime factorization of n be a, and the exponent of 3 be b. Thus 2 contributes a factor of $a + 1$ to $d(n)$, and 3 contributes a factor of $b + 1$ to $d(n)$.

The exponent of 2 in the prime factorization of $2n$ is $a + 1$, so 2 contributes a factor of $(a + 1) + 1 = a + 2$ to $d(2n)$. Thus we have

$$\frac{8d}{6d} = \frac{4}{3} = \frac{a+2}{a+1}$$

which means $a = 2$.

Similarly, if we analyze the exponent of 3, we get

$$\frac{9d}{6d} = \frac{3}{2} = \frac{b+2}{b+1}$$

thus $b = 1$.

Therefore $n = 2^2 \cdot 3 \cdot k$ for some positive integer k. Importantly, k cannot contain any factors of 2 or 3 (as that would change a and b). Notice that $12k \leq 150$ which means $k \leq 12$. Thus k can be $1, 5, 7, 11$ so the answer is $12(1+5+7+11) = \boxed{288}$.

15. **[10]** Lewis is standing at the origin of the xy-plane. At any given moment, he has a $1/2$ chance of moving 1 unit upward and a $1/2$ chance of moving 1 unit to the right. Given that he never reaches the point $(3, 3)$, what is the probability that Lewis will reach the point $(4, 4)$?

Proposed by Lewis Lau.

Answer: $\boxed{\frac{15}{88}}$

Solution: Consider Lewis's location after 6 movements. With $\binom{6}{i}$ paths to $(i, 6-i)$, there is a $\binom{6}{i}/64$ chance Lewis is at $(i, 6-i)$. For Lewis to reach $(4, 4)$, he must not be at any of $(6, 0)$, $(5, 1)$, $(1, 5)$, or $(0, 6)$. Furthermore, we know he can't reach $(3, 3)$. Thus, he must be at either $(4, 2)$ or $(2, 4)$.

In the case of $(4, 2)$, he must move up twice. Otherwise, he will be too far to the right to ever reach $(4, 4)$. Thus, there is a $\binom{6}{2}/64/4 = 15/256$ chance to reach $(4, 2)$ and $(4, 4)$. By symmetry, the probability is the same for $(2, 4)$. Since the chance of reaching $(3, 3)$ is $\binom{6}{3}/64 = 20/64$, the final answer is

$$\frac{2 \cdot \frac{15}{256}}{1 - 20/64} = \boxed{\frac{15}{88}}.$$