MBMT Team Round — Leibniz

March 30, 2019

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **15** questions. You will have **45** minutes to complete the round. Later questions are worth more points; point values are notated next to the problem statement. (There are a total of 100 points.) Please write your answers in the simplest possible form.

DO NOT TURN THE QUESTION SHEET IN!
Use the official answer sheet.

You are highly encouraged to work with your teammates on the problems in order to solve them.

____________ **1** **[4]** At an interesting supermarket, the nth apple you purchase costs n dollars, while pears are 3 dollars each. Given that Layla has exactly enough money to purchase either k apples or $2k$ pears for $k > 0$, how much money does Layla have?

____________ **2** **[4]** Real numbers a, b, c are selected uniformly and independently at random between 0 and 1. What is the probability that $a \geq b \leq c$?

____________ **3** **[4]** For how many positive integers $1 \leq n \leq 10$ does there exist a prime p such that the sum of the digits of p is n?

____________ **4** **[5]** What is the maximum number of intersection points between 2 circles and 2 triangles?

____________ **5** **[5]** There are 50 dogs in the local animal shelter. Each dog is enemies with at least 2 other dogs. Steven wants to adopt as many dogs as possible, but he doesn't want to adopt any pair of enemies, since they will cause a ruckus. Considering all possible enemy networks among the dogs, find the maximum number of dogs that Steven can possibly adopt.

____________ **6** **[5]** How many ordered pairs of positive integers (x, y), where x is a perfect square and y is a perfect cube, satisfy $\operatorname{lcm}(x, y) = 81000000$?

____________ **7** **[6]** Unit circles a, b, c satisfy $d(a, b) = 1$, $d(b, c) = 2$, and $d(c, a) = 3$, where $d(x, y)$ is defined to be the minimum distance between any two points on circles x and y. Find the radius of the smallest circle entirely containing a, b, and c.

____________ **8** **[6]** The numbers 1 through 5 are written on a chalkboard. Every second, Sara erases two numbers a and b such that $a \geq b$ and writes $\sqrt{a^2 - b^2}$ on the board. Let M and m be the maximum and minimum possible values on the board when there is only one number left, respectively. Find the ordered pair (M, m).

____________ **9** **[7]** N people stand in a line. Bella says, "There exists an assignment of nonnegative numbers to the N people so that the sum of all the numbers is 1 and the sum of any three consecutive people's numbers does not exceed $1/2019$." If Bella is right, find the minimum value of N possible.

____________ **10 [8]** In triangle $\triangle ABC$, D is on AC such that BD is an altitude, and E is on AB such that CE is an altitude. Let F be the intersection of BD and CE. If $EF = 2FC$, $BF = 8DF$, and $DC = 3$, then find the area of $\triangle CDF$.

____________ **11 [8]** Consider nonnegative real numbers $a_1, \ldots, a_6$ such that $a_1 + \cdots + a_6 = 20$. Find the minimum possible value of

$$\sqrt{a_1^2 + 1^2} + \sqrt{a_2^2 + 2^2} + \sqrt{a_3^2 + 3^2} + \sqrt{a_4^2 + 4^2} + \sqrt{a_5^2 + 5^2} + \sqrt{a_6^2 + 6^2}.$$

____________ **12 [9]** Given two points A and B in the plane with $AB = 1$, define $f(C)$ to be the incenter of triangle ABC, if it exists. Find the area of the region of points $f(f(X))$ where X is arbitrary.

____________ **13 [9]** Find an $a < 1000000$ so that both a and $101a$ are triangular numbers. (A triangular number is a number that can be written as $1 + 2 + \cdots + n$ for some $n \geq 1$.)

Note: There are multiple possible answers to this problem. You only need to find one.

____________ **14 [10]** Leptina and Zandar play a game. At the four corners of a square, the numbers 1, 2, 3, and 4 are written in clockwise order. On Leptina's turn, she must swap a pair of adjacent numbers. On Zandar's turn, he must choose two adjacent numbers a and b with $a \geq b$ and replace a with $a - b$. Zandar wants to reduce the sum of the numbers at the four corners of the square to 2 in as few turns as possible, and Leptina wants to delay this as long as possible. If Leptina goes first and both players play optimally, find the minimum number of turns Zandar can take after which Zandar is guaranteed to have reduced the sum of the numbers to 2.

____________ **15 [10]** There exist polynomials P, Q and real numbers $c_0, c_1, c_2, \ldots, c_{10}$ so that the three polynomials

$$P, \quad Q, \text{ and } \quad c_0P^{10} + c_1P^9Q + c_2P^8Q^2 + \cdots + c_{10}Q^{10}$$

are all polynomials of degree 2019. Suppose that $c_0 = 1, c_1 = -7, c_2 = 22$. Find all possible values of c_{10}.

Note: The answer(s) are rational numbers. It suffices to give the prime factorization(s) of the numerator(s) and denominator(s).