MBMT Geometry Round — Gauss

April 7, 2018

Full Name ____

Team Number __________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO DO SO.

This round consists of **8** questions. You will have **30** minutes to complete the round. Each question is *not* worth the same number of points. Questions answered correctly by fewer competitors will be weighted more heavily. Please write your answers in a reasonably simplified form.

1 27 cubes of side length 1 are arranged to form a $3 \times 3 \times 3$ cube. If the corner $1 \times 1 \times 1$ cubes are removed, what fraction of the volume of the big cube is left?

Proposed by Chris Tong

Solution. $\boxed{\dfrac{19}{27}}$

There are 8 corners. If the 8 corner pieces are removed, then the resulting solid will contain $27 - 8 = 19$ 1 by 1 by 1 cubes. Therefore, the ratio of the volumes is $\boxed{\dfrac{19}{27}}$. $\square$

2 Points A, B, and C are on a line such that $AB = 6$ and $BC = 11$. Find all possible values of AC.

Proposed by Daniel Zhu

Solution. $\boxed{5, 17}$

Depending on the location of A, AC is either $11 + 6 = 17$ or $11 - 6 = 5$. $\square$

3 Consider rectangle $ABCD$, with $1 = AB < BC$. The angle bisector of $\angle DAB$ intersects $\overline{BC}$ at E and $\overrightarrow{DC}$ at F. If $FE = FD$, find BC.

Proposed by Steven Qu

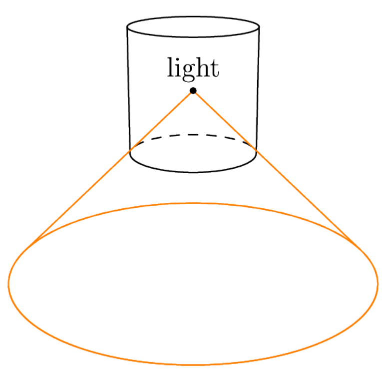
Solution. $\boxed{2 + \sqrt{2}}$

Let $BC = x$.

$$FE = FA - EA = (x - 1)\sqrt{2}, FD = AD = x.$$

Equating and solving yields $x = \boxed{2 + \sqrt{2}}$. $\square$

4 Consider a lamp in the shape of a hollow cylinder with the circular faces removed with height 48 cm and radius 7 cm. A point source of light is situated at the center of the lamp. The lamp is held so that the bottom of the lamp is at a height 48 cm above an infinite flat sheet of paper. What is the area of the illuminated region on the flat sheet of paper, in cm^2?



Proposed by David Wu

Solution. $\boxed{441\pi}$

Take a cross section so that the paper is a line ℓ and the lamp is a rectangle $ABCD$ (labelled counterclockwise) with dimensions 14 by 96 with $CD \parallel \ell$ and CD closer to ℓ than AB. Let M be the midpoint of CD, and take the point source in the rectangle to be O and extend OC to meet ℓ at X and OM to meet ℓ at Y. Then we have two similar right triangles: OCM and OXY. Note that $OM = 24$, $CM = 7$, and $MY = 72$. Hence, by similar triangles, $\frac{7}{XY} = \frac{24}{24+48} \implies XY = 21$.

If we imagine tracing light rays from O to the paper, we see that the rays intersect the paper with the silhouette of a circle with radius XY. So the area is $XY^2\pi = \boxed{441\pi}$, as desired. $\square$

5 There exist two triangles ABC such that $AB = 13$, $BC = 12\sqrt{2}$, and $\angle C = 45°$. Find the positive difference between their areas.

Proposed by Steven Qu

Solution. $\boxed{60}$

Let A and A' be the two possible locations for A if we fix B and C as well as the direction of AC. Let BD be the height to side AC. Then $BD = 12$ by 45-45-90 triangles. Then both ADB and $A'DB$ are 5-12-13 triangles. Therefore, the different of the areas is just two $5-12-13$ triangles, which have a total area of 60 $\square$

6 $\triangle ABC$ is a right triangle with $\angle A = 90°$. Square $ADEF$ is drawn, with D on $\overline{AB}$, F on $\overline{AC}$, and E inside $\triangle ABC$. Point G is chosen on $\overline{BC}$ such that EG is perpendicular to BC. Additionally, $DE = EG$. Given that $\angle C = 20°$, find the measure of $\angle BEG$.

Proposed by Kevin A. Zhou

Solution. $\boxed{55°}$

Since the angles of a triangle add up to $180°$, the $\angle B = 70°$. Notice that point E is the incenter of triangle ABC. This means that BE bisects $\angle ABC$, so the measure of $\angle GBE = 35°$. Then, the measure of $\angle BEG = \boxed{55°}$. $\square$

7 Let ABC be an equilateral triangle with side length 2. Let the circle with diameter AB be Γ. Consider the two tangents from C to Γ, and let the tangency point closer to A be D. Find the area of $\triangle CAD$.

Proposed by Steven Qu

Solution. $\boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}}$

$[CAD] = [OAD] + [OCD]\check{\ }[OAC]$. We tackle the triangles on the RHS one by one.

$OC = \sqrt{3}$, so $[OAC] = \frac{\sqrt{3}}{2}$. Furthermore, OD $= 1$ and CD is tangent to O, so OD is perpendicular to CD and $CD = \sqrt{2}$. This means that $[OCD] = \frac{\sqrt{3}}{2}$. Now, drop a perpendicular from D to AB, call the foot E. DE is parallel to OC, so $\angle ODE = \angle DOC$ and triangles OED and CDO are similar. Therefore, $DE = \frac{\sqrt{3}}{3}$, and $[OAD] = \frac{\sqrt{3}}{6}$. Thus, the answer is $\frac{\sqrt{3}}{6} + \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}$.

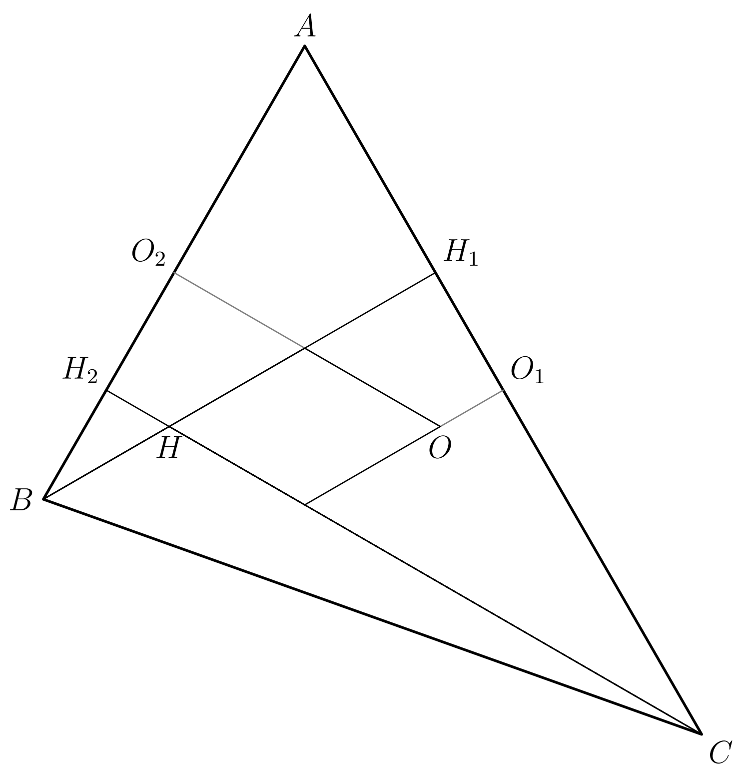
Diagram(s) here: (diagram 2 for problem, diagram 3 for solution) https://tinyurl.com/mbmtdiagrams □

8 Let ABC be a triangle with $\angle A = 60^\circ$, $AB = 37$, $AC = 41$. Let H and O be the orthocenter and circumcenter of ABC, respectively. Find OH.

The orthocenter of a triangle is the intersection point of the three altitudes. The circumcenter of a triangle is the intersection point of the three perpendicular bisectors of the sides.

Proposed by Daniel Zhu

Solution. $\boxed{4}$



Let H_1, H_2 be the projections of H onto AC and AB. Let O_1, O_2 be the projections of O onto AC and AB.

Note that $H_1O_1 = AO_1 - AH_1 = \frac{41}{2} - \frac{37}{2} = 2$. Similarly, $O_2H_2 = 2$.

Through some 30-60-90 triangles, we get that the quadrilateral formed by the four perpendiculars is a rhombus with side length $\frac{4}{\sqrt{3}}$ and angles of 60° and 120°. Therefore we can compute OH to be $\boxed{4}$. □